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The Tides in the Labrador Sea, Davis Strait and Baffin Bay

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1966

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by Gabriel Godin

ABSTRACT

The tides in the sea made up of the Labrador Sea, Davis Strait and Baffin Bay, are investigated theoretically. It is shown that the tidal regime is controlled mainly by the oscillations of the Atlantic.

One dimensional calculations for M_2 and K_1, the main semidiurnal and diurnal constituents, yield profiles of the elevation along the sea which compare well with observations once proper account is taken of the diffuse reflection which takes place at the head. Currents of the order of 10 cm/sec are calculated for M_2 at the mouth of the Sea of Labrador, while the currents due to K_1 are insignificant everywhere in the sea except in Davis Strait. The maximum semidiurnal streams as well are found in Davis Strait where they are of the order of 15 cm/sec.

Two dimensional calculations allow us to take account of the earth's rotation. In order to carry these out, the sea is schematized into three basins of constant width and depth. With such a schematization it is possible to make some headway into the calculations once the concept of a narrow deep elongated sea is introduced. Such a sea is the intermediate between an infinite channel of constant depth and a broad rectangular sea of constant depth. As a consequence it will be shown that the motion in such a basin is controlled mainly by Kelvin waves and that the Poincaré waves are important only in the immediate vicinity of the boundaries. The motion is such a sea is described in a series of theorems. Once these are established it is possible to obtain a solution for M_2 for the model. A solution for K_1 fails to exist due to divergences at the inner boundaries.

Using the solution for M_2, cotidal charts for the elevation and the transport per unit width are drawn as well as a map of the elements of the current ellipses. The agreement between the cotidal chart for the elevation derived from coastal elevations and the one derived from the theoretical calculations is most satisfactory considering the crudeness of the model used.

It is established that the motion at the mouth of the Sea of Labrador is purely of the Kelvin type. Over most of the sea, the motion is completely described by the superposition of two standing Kelvin waves.

NOTATION

The quantities which have dimension are followed by their dimensions in brackets.

M: mass (grams) L: length (centimetres) T: time (seconds)

a	Half width of a rectangular basin (L)
a_j	Vector component (always indexed)
A	Arbitrary constant (L, LT^{-1})
$A_\pm$	Column vector of arbitrary constants (LT^{-1})
A_0	Arbitrary constant pertaining to a Kelvin wave (LT^{-1})
b	Width of a channel (L, LT^{-1})
$\underline{b}_j$	Basic vector
B	Arbitrary constant (L, LT^{-1})
B_0	Arbitrary constant pertaining to a Kelvin wave (LT^{-1})
c_j	Vector component
C	Arbitrary constant (LT^{-1})
D	Arbitrary constant (LT^{-1})
e	Base of the natural logarithms
f	Function
$\underline{f}$	Function denoted as a vector
F	Gravitational force (MLT^{-2})
g	Acceleration due to gravity (LT^{-2})
h	Depth (L)
i	Imaginary unit ($i^2=-1$)
I	Imaginary part of. Thus A^I: the imaginary part of A

j	Any positive integer
j_k	A positive integer belonging to a set k of positive integers
$j' \equiv \frac{j\pi}{2a}$	(L^{-1})
$J_r(x)$	Bessel function of the first kind of order r and argument x
k	A positive integer being either 1 or 3
$k^2 \equiv \frac{\sigma^2 - 4\omega^2}{gh}$	(L^{-2})
$K \equiv \sigma/\sqrt{gh}$	(L^{-1})
$K'_0 \equiv \sigma/\sqrt{gh}$	(L^{-1})
$K_0 \equiv \left(\sigma/\sqrt{gh}\right) \cdot \frac{2a}{\pi}$	
$K_j \equiv \left\{ j^2 - \frac{\sigma^2 - 4\omega^2}{gh} \cdot \left(\frac{2a}{\pi}\right)^2 \right\}^{\frac{1}{2}}$	
$l \equiv \frac{\pi L}{2a}$	Dimensionless equivalent of length
L	1) Length of a rectangular basin (L) 2) Latitude
m	Odd positive integer
M	Odd positive integer (distinct from m)
n	Even positive integer
N	Even positive integer (distinct from n)
P_n	Vector component of $\cosh \lambda' x$ in the direction of the basic vector $\cos n'x$
$P_{j_k n_i}$	Vector component of $\cos j'_k x$ in the direction of the basic vector $\cos n'_i x$ (When $i \equiv k$ the subindices i and k are dropped)
q_m	Vector component of $\sinh \lambda' x$ in the direction of the basic vector $\sin m'x$
$Q_{j_k m_i}$	Vector component of $\sin j'_k x$ in the direction of the basic vector $\sin m'_i x$
r	Order of a Bessel function

R	Real part of. Thus A^R : the real part of A
t	Time (T)
u	Velocity component along the x direction (LT^{-1})
v	Velocity component along the y direction (LT^{-1})
$\underline{V}$	A vector
x	Distance along one direction of the Cartesian axes (L); $\frac{2\sigma}{\propto \sqrt{gh}}$
y	Distance along a direction of the Cartesian axes perpendicular to x (L)
X	A function dependant only on the x variable; or $e^{-\propto y}$
Y	A function dependant only on the y variable
Z	Elevation above mean level
α	Exponential increase in depth per unit length (L^{-1})
$\alpha \equiv \frac{2a}{\pi g}$	(T^2)
β	Exponential increase in width per unit length (L^{-1})
$\Gamma_\pm$	Matrices of coefficients
δ_{ij}	Kronecker delta symbol $\begin{cases} = 1 & i = j \\ = 0 & i \neq j \end{cases}$
$\delta_\pm$	Column vectors of constant terms
$\delta_\pm$	Column vectors of the coefficients of B_0
$\zeta_\pm$	Column vectors of the coefficients of (C_1, D_1)
θ_j	Vector component of v at the mouth of the sea in the direction of the basic vectors $\left(\begin{matrix}\sin m'x \\ \cos n'x\end{matrix}\right)$
$\lambda \equiv 2\omega/\sqrt{gh}$	(L^{-1})
$\lambda' \equiv \left(2\omega/\sqrt{gh}\right) . \left(\frac{2a}{\pi}\right)$	
$\wedge$	Wave length (L)

σ	Angular speed (T^{-1})
Σ	Summation sign
ω	Angular speed of the earth's rotation (T^{-1})
Ω	$\equiv \frac{2\omega\sigma_1}{gh}\left(\frac{2a}{\pi}\right)^2$ A small quantity for a narrow deep elongated sea
$\sim$	Of the order of
$\simeq$	Approximately equal to
$\equiv$	Is defined by or is identical to
$\longrightarrow$	Goes into or tends towards
(-)	(-1)

Double subscripts are used in the latter part of the thesis. These are necessary to indicate to which set of indices a given index belongs. Thus A_{nk} denotes an arbitrary constant affected by an even index belonging to a set k.

Overscripts as well are used and are circled to distinguish them from exponents.

PRELIMINARIES

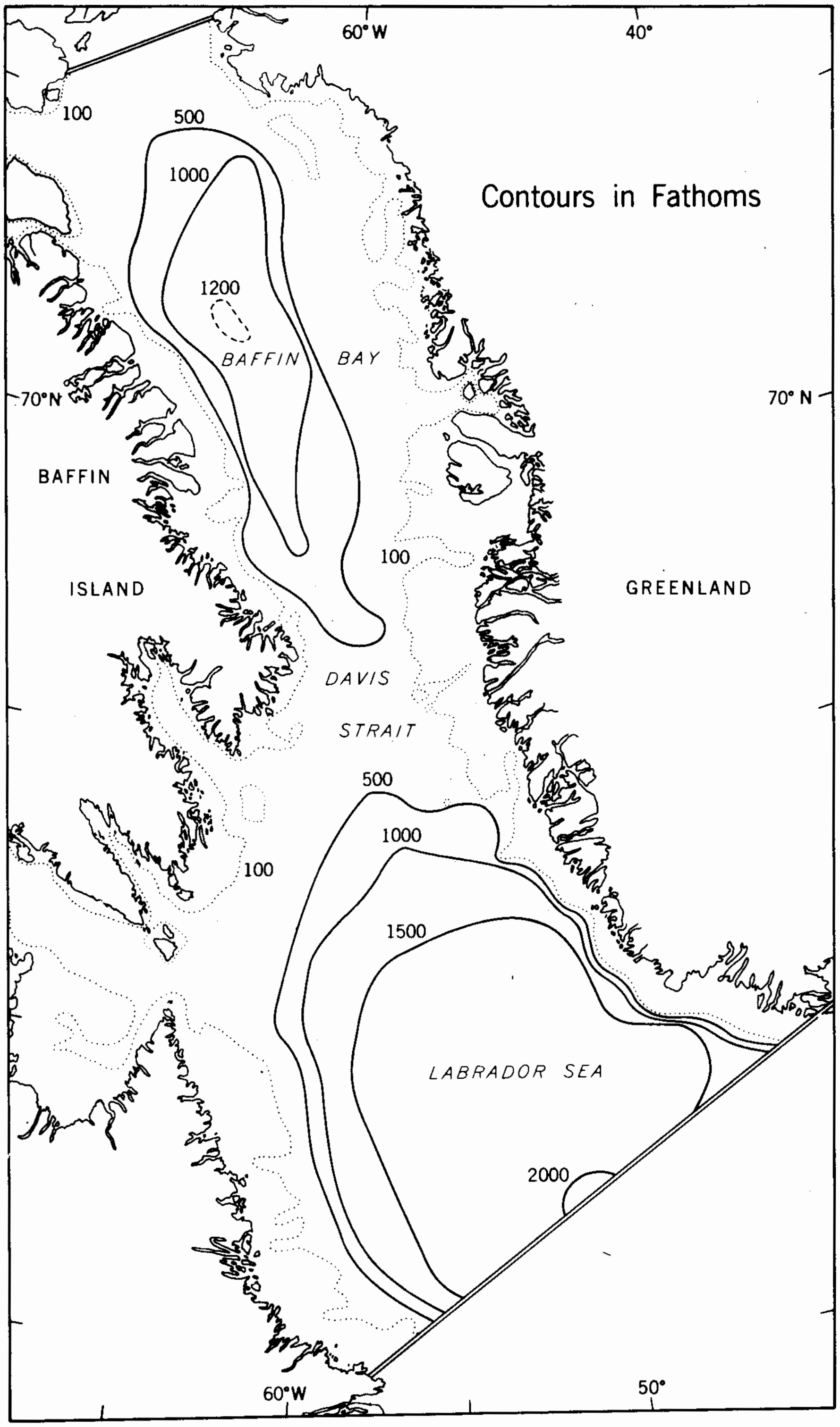

60° W
40°
Contours in Fathoms
100
500
1000
1200
BAFFIN BAY
70°N
70° N
BAFFIN
ISLAND
GREENLAND
100
DAVIS
STRAIT
500
1000
100
1500
LABRADOR SEA
2000
60°W
50°

Fig. 1

Physical and Tidal Characteristics of the Area Under Study

The Labrador Sea, Davis Strait and Baffin Bay are part of a single sea depicted in Fig. 1. The double lines indicate the boundaries chosen for this sea. The openings of Hudson Strait, Lancaster Sound, Jones Sound and Smith Sound denoted by the letters A B C D are narrow and shallow compared to the main body of the sea so that we may consider this one as approximately closed in its sides and northern boundary.

The Labrador Sea, Davis Strait and Baffin Bay form the three natural divisions of the domain under study; the first being broad and deep, the second narrow and shallow and the third being a bay insofar as it is connected to the outside ocean by a gully represented by Davis Strait.

The distance between the two boundaries shown in Fig. 1 is approximately 2350 km. The width varies between 970 km at the mouth to 300 km at the narrowest point in Davis Strait. If we transform the actual cross sections of the sea into rectangular sections of constant depth (the mean depth of the section), the depth of such sections varies from 2600 m at the mouth to 310 m at the shallowest point in Davis Strait located at 1140 km from the mouth. The greatest mean depth in Baffin Bay is 1100 m to be found at about 1800 km from the mouth. Fig. 4 shows the variation of width and mean depth along the sea along with an attempt of schematization will be discussed later. Fig. 4 is to be found on p. 25.

In spite of adverse climatic and geographic conditions, tidal information on this area has been slowly accumulating over the years and it is now possible to have a fair idea of the tidal regime prevailing there. Most of the observations however cover only short periods; the slow constituents therefore are not well known and our attention will be concentrated on the diurnal and semi diurnal tides. Tables 1 and 2 that follow give the data available: Table 1 gives the name and location of the station indicated by a number in Fig. 2 and 3 while Table 2 gives the pertaining tidal data in Atlantic Standard Time. Fig. 2 and 3 show cotidal charts for M_2 and K_1 based on such observations.

Table I
Tidal Stations

1.	Point Aldrich Cape Columbia, Grant Land	Lat. 83° 07'N Long. 69° 40'W 29 days 1908 U. S. Coast & Geod. Surv.
2.	Alert Ellesmere Island	Lat. 82° 30'N Long. 62° 20'W 2 x 29, 1 x 15 days 1960 C. H. S.
3.	Cape Sheridan Grant Land	Lat. 82° 27'N Long. 61° 30'W
4.	Fort Conger Discovery Harbour	Lat. 81° 44'N Long. 64° 44'W 369 days 1881 - 1882 U. S. Coast & Geod. Surv.
5.	Dundas Harbour Lancaster Sound	Lat. 74° 31'N Long. 82° 26'W 29 days Sept. 1960 C. H. S.
6.	Radstock Bay Devon Island	Lat. 72° 42'N Long. 91° 05'W 15 days Sept. 1961 C. H. S.
7.	Cape Hooper Baffin Island	Lat. 68° 21'N Long. 66° 45'W 30 days Aug. 1958 C. H. S.
8.	Kivitoo Baffin Island	Lat. 67° 56'N Long. 64° 56'W 30 days Aug. 1959 C. H. S.
9.	Broughton Island Davis Strait	Lat. 67° 31'N Long. 64° 04'W 15 days 1958 C. H. S.
10.	Kingua Fjord Cumberland Sound Davis Strait	Lat. 66° 30'N Long. 67° 20'W 41 days 1883 International Polarforschung
11.	Cape Dyer Baffin Island	Lat. 66° 33"N Long. 61° 39'W 30 days Sept. 1958 C. H. S.

12.	Brevoort Harbour Davis Strait	Lat. 63° 19'N Long. 64° 09'W 29 days 1957 C. H. S.
13.	Koojesse Inlet Frobisher Bay	Lat. 63° 43'N Long. 68° 41'W 30 days 1958 15 days 1959 C. H. S.
14.	Lewis Bay Frobisher Bay	Lat. 63° 20'N Long. 67° 57'W 29 days Oct. 1959
15.	Whiskukun Frobisher Bay	Lat. 63° 13'N Long. 68° 03'W 30 days Sept. 1958 C. H. S.
16.	Frobisher Farthest Frobisher Bay	Lat. 63° 29'N Long. 68° 02'W 29 days 1951 C. H. S.
17.	Tanner Bay Resolution Island	Lat. 61° 37'N Long. 64°44'W 2 days Sept. 1952 C. H. S.
18.	Port Burwell Hudson Strait	Lat. 60° 25'N Long. 64° 52'W 15 days 1885 Proc. Roy. Soc. London 45 1888-89
19.	Williams Harbour Labrador	Lat. 60° 00'N Long. 64° 19'W 15 days 1952 C. H. S.
20.	Nain Labrador	Lat. 56° 33'N Long. 61° 45'W 29 days 1932 Admiralty
21.	Hopedale Labrador	Lat. 55° 07'N Long. 60° 13'W 15 days 1959 C. H. S.
22.	Makkovik Labrador	Lat. 55° 06'N Long. 59° 10'W 30 days 1958 C. H. S.

23.	Cartwright Labrador	Lat. 53° 42'N Long. 57° 02'W 29 days 1934 Admiralty
24.	Cape Bryant Greenland	Lat. 82° 21'N Long. 54° 30'W 29 days 1909 U. S. Coast & Geod. Surv.
25	Thank God Harbour Polaris Bay, Greenland	Lat. 81° 36'N Long. 61° 40'W 174 days U. S. Coast & Geod. Surv.
26.	Rensselaer Greenland	Lat. 78° 37'N Long. 70° 53'W 116 days 1853-54 U. S. Coast & Geod. Surv.
27.	Port Foulke Greenland	Lat. 78° 18'N Long. 73° 00'W 58 days 1860-61 U. S. Coast & Geod. Surv.
28.	Thule Greenland	Lat. 76° 32'N Long. 68° 54'W
29.	Kamarajuk Fjord Umanak District	Lat. 71° 00'N Long. 51° 00'W 2 x 15 days 1930 Wegener
30.	Godhavn Disko Island, Greenland	Lat. 69° 15'N Long. 53° 33'W 29 days 1936 Danish Hydrographic Office
31.	Nunarssuaq Kronprinsens Ejland, Greenland	Lat. 68° 59'N Long. 53° 21'W 3 x 29 days 1950 Danish Hydrographic Office
32.	Egedesminde Greenland	Lat. 68° 43'N Long. 52° 53'W 29 days 1952 Danish Hydrographic Office
33.	Aningaq Rifkol, Greenland	Lat. 67° 55'N Long. 53° 50'W 3 x 29 days 1950 Danish Hydrographic Office
34.	Godthaab Greenland	Lat. 64° 11'N Long. 51° 45'W 145 days 1951 Danish Hydrographic Office

35.	Faeringehavn Greenland	Lat. 63° 42'N Long. 51° 33'W 3 x 15 days 1952 Danish Met. Office
36.	Grønnedal Greenland	Lat. 61° 13'N Long. 48° 07'W 29 days 1958 Danish Hydrographic Office
37.	Julianehaab Greenland	Lat. 60° 43'N Long. 46° 02'W 145 days 1934 J. Egedal J. Thule Expedition
38.	Nannortalik Greenland	Lat. 60° 10'N. Long. 45° 20'W

Table II

ZONE +0400

STN	K1		O1		P1		M2		S2		N2	
	Z CM	G °	Z CM	G °	Z CM	G °	Z CM	G °	Z CM	G °	Z CM	G °
1	5	321	3	295	2	321	12	254	5	304	2	222
2	5	287	3	261	2	287	22	278	11	321	3	342
3	5	299	3	285	2	297	25	310	12	354	4	285
4	9	226	3	208	2	238	60	348	27	028	12	325
5	34	203	11	170	11	208	66	022	26	060	14	348
6	27	243	17	205	9	243	63	013	26	049	11	001
7	19	191	13	142	6	191	20	122	6	180	3	089
8	21	189	6	145	7	189	247	138	11	186	5	088
9	25	189	8	123	8	189	31	138	13	180	5	116
10	8	039	3	050	3	039	227	178	81	217	37	144
11	11	196	3	132	4	096	84	157	25	199	13	117
12	17	064	10	029	6	064	181	196	64	225	35	161
13	18	123	10	063	6	123	334	221	115	270	61	192
14	22	126	10	055	7	126	294	228	110	277	72	201
15	20	111	9	063	6	111	334	226	119	272	67	201
16	18	101	9	062	6	098	321	219	110	267	54	189
17	32	101	11	070			175	214	58	257		
18	15	119	6	166	5	119	217	277	71	305	20	323
19	19	096	10	032	6	095	98	231	24	272	30	197
20	12	129	8	102	4	129	75	199	22	227	15	188
21	8	115	7	071	3	107	62	192	23	214	14	177
22	10	116	7	095	3	116	56	190	21	222	12	355
23	11	120	6	065	4	120	53	198	20	231	11	180
24	10	281	4	261	3	286	13	357	7	036	2	347

25	12 248	5 215	4 246	55 359	25 036	11 337
26	26 204	13 165	9 204	103 359	46 034	21 333
27	32 200	12 163	11 200	111 352	46 031	20 327
28	40 196	12 154		79 339	31 012	15 316
29	37 139	9 134	12 139	46 284	17 302	11 282
30	32 161	13 130	10 161	60 254	25 288	17 250
31	35 168	12 125	11 168	57 260	27 295	17 225
32	36 148	11 126	12 148	67 235	26 265	12 210
33	36 160	9 107	12 160	78 235	32 262	16 217
34	20 114	10 077	7 114	138 176	57 210	28 156
35	18 110	10 086	6 110	117 174	40 230	18 142
36	16 104	10 063	5 104	93 162	35 196	14 133
37	16 098	9 063	5 098	87 180	33 182	17 128
38	19 099	11 060		88 142	38 173	

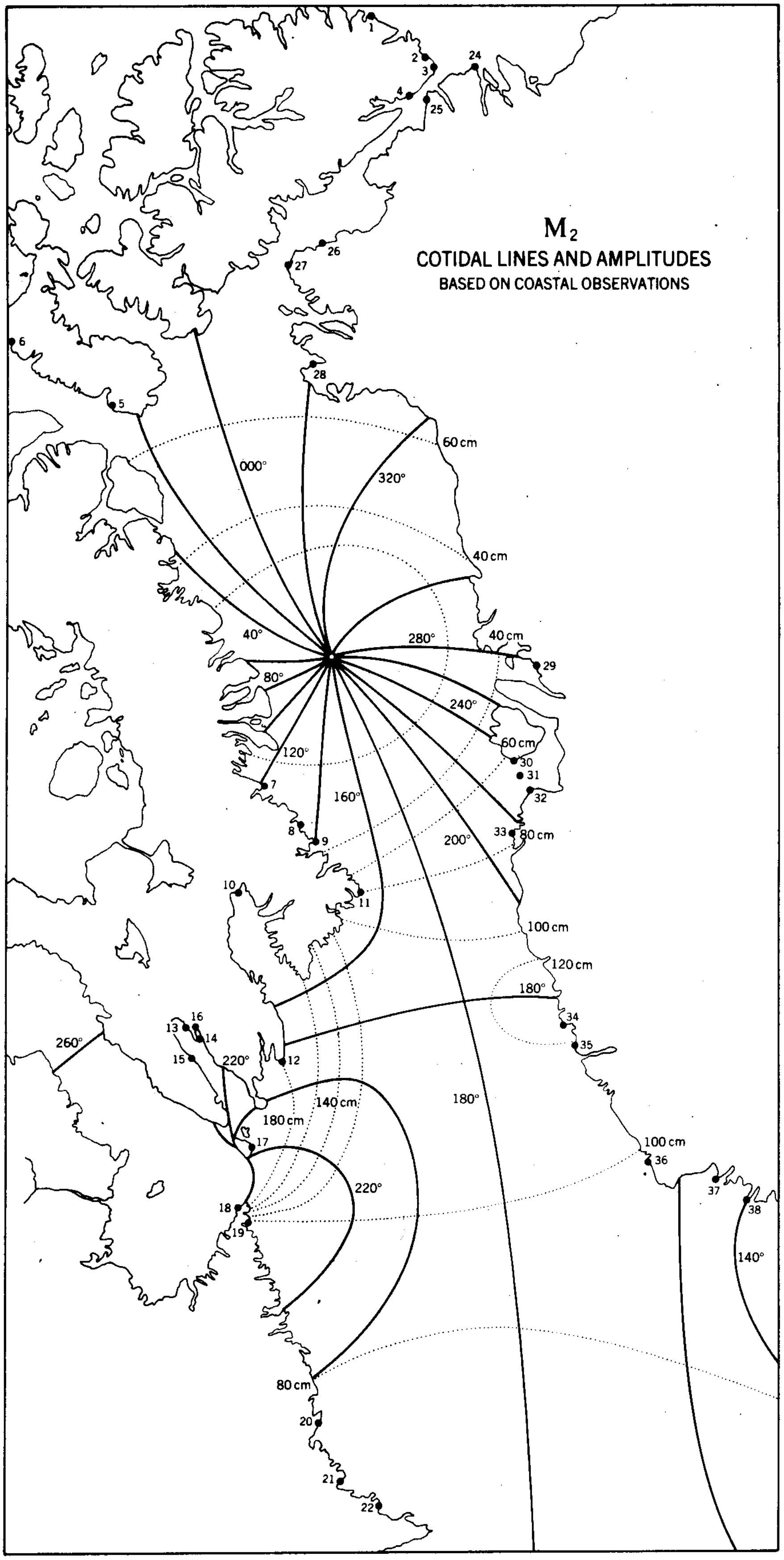
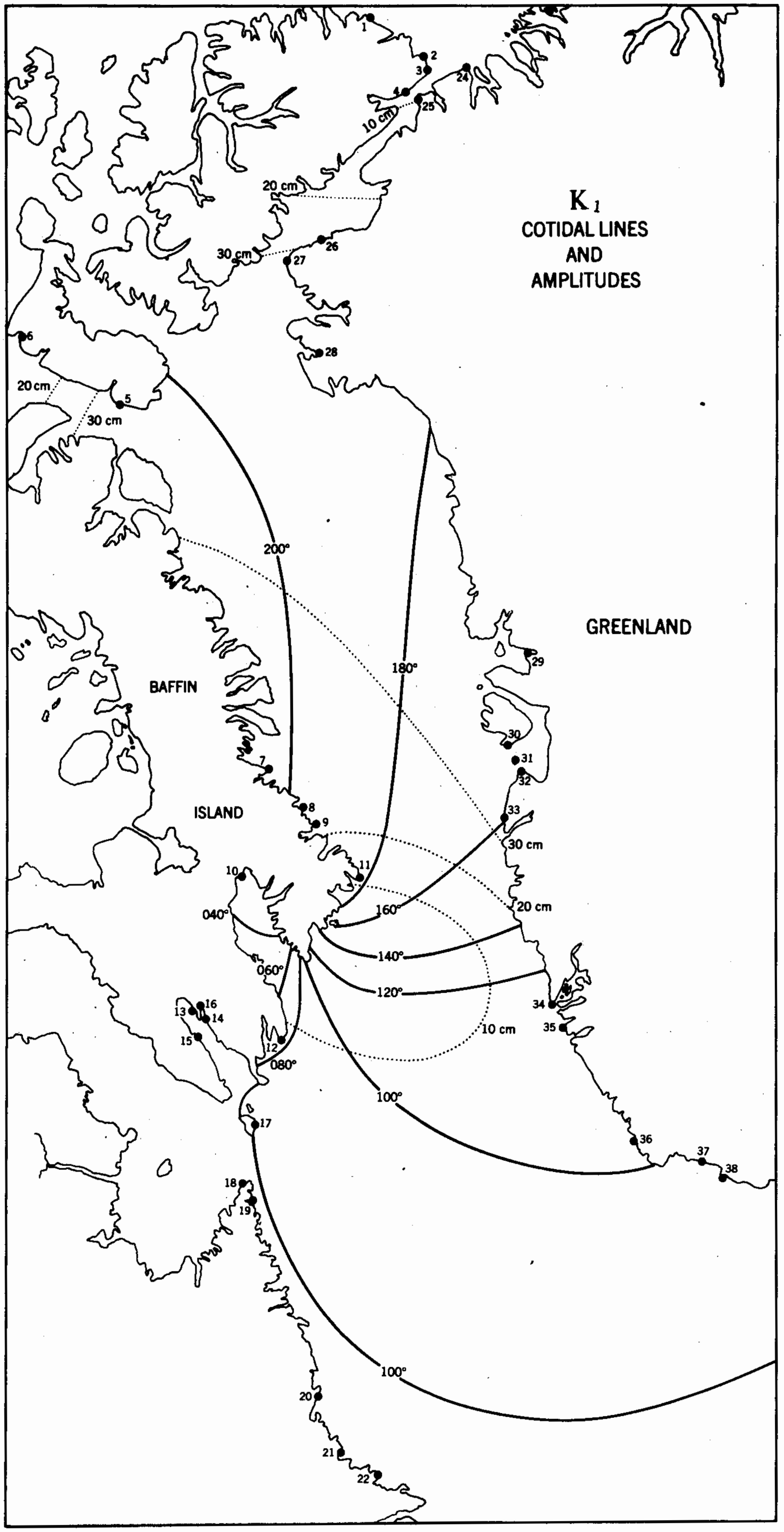

M_2
COTIDAL LINES AND AMPLITUDES
BASED ON COASTAL OBSERVATIONS
1
2
3
24
4
25
26
27
6
28
5
60 cm
000°
320°
40 cm
40 cm
40°
280°
29
80°
240°
60 cm
120°
30
31
7
160°
32
8
9
33
80 cm
200°
10
11
100 cm
120 cm
180°
34
260°
13
16
14
35
15
220°
12
140 cm
180°
180 cm
17
100 cm
36
37
220°
38
18
19
140°
80 cm
20
21
22

Fig. 2


K_1
COTIDAL LINES
AND
AMPLITUDES
GREENLAND
BAFFIN
ISLAND
200°
180°
160°
140°
120°
100°
100°
040°
060°
080°
10 cm
20 cm
30 cm
20 cm
30 cm
30 cm
20 cm
10 cm
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
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33
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36
37
38

Fig. 3

The remarkable features of the M_2 tide are:

1. - The large amplitudes on the west coast in the vicinity of Hudson Strait,

2. - The presence of a point of amphidromy in the vicinity of latitude 72° 45'N.

Those of the K_1 tide are:

1. - The presence of a maximum in amplitude near the head of the sea,

2. - A degenerate point of amphidromy in the vicinity of 66° N.

N_2 , S_2 on one hand and O_1 on the other hand, exhibit the same characteristics; the position of their points of amphidromy varies slightly from that of M_2 and K_1 .

Review of the Literature

Harris (7) and Sterneck (14) gave some cursory comments about the tides prevailing in the area under study. The observational material was then very scanty and they had to limit themselves to generalities. Harris states that the tidal wave is of a standing character; the contraction of Davis Strait suggests to him that the cotidal lines there are crowded, which is not the case. Sterneck as well draws a set of cotidal lines across the sea.

Defant (2), in need of a value for the transport at the mouth of the sea of Labrador in his study of the tides of the Atlantic, applied his canal theory to this area which was eminently suitable for such an approach, and he obtained remarkably good results for M_2 . He located the node at the very location where subsequent observations would place it and he indicated that the maximum amplitude of $M_2$was to be found ahead of the contraction of Davis Strait which again conforms strictly to reality.

Recently, the two Russian workers Al'tschuler and Vladimirov (1) applied Hansen's (6) (so called Polukarov's) method for the calculation of the M_2 tide from the boundary values to the same area. This method represents one of the most significant advances in this field of knowledge; on the other hand its success depends on the quality of the boundary values. In this instance the Russian workers seem to have interpolated values for their stations #23 (70° 22'N, 67° 52'W) and #24 (71° 33'N, 71° 30'W) which lie on Baffinland from stations lying on Greenland. This had disastrous consequences since the two land masses lie across a point of amphidromy and there is a very marked change of phase from one side to the other. The results of their calculations reflect exactly the features they had introduced into their data, this is to say, a set of cotidal lines lying across the sea.

In our work we shall study the tidal motion in channels and rotating seas. The equations of tidal motion in channels without friction are simple and such a type of motion is adequately surveyed in Lamb (9) who describes in particular the work of McCowen (10) and Green (4). The form of the solution varies with the law assumed for the variation in width and depth.

Motion in rotating seas is analytically much more difficult and we restrict ourselves to rectangular seas of constant depth. In this case the equations can be solved with the help of eigenfunctions. Using the standard apparatus of vector spaces it is possible to investigate the motion in such a sea.

The study of tidal motion in rotating rectangular seas of constant depth was initiated by Thomson (16) and Poincaré (11). In 1921 Taylor (15) solved the problem of the reflection of a Kelvin wave in a semi infinite rectangular sea of constant depth; he showed that both the Kelvin and the Poincaré waves are necessary to give a complete solution and that only by assuming their simultaneous presence is it possible to satisfy any boundary condition. In 1931 Grace (3) introduced some simplification into the solution given by Taylor by assuming the smallness of $\left(\frac{\sigma\alpha}{gh}\right)^2$

and he showed that the effect of the boundary value at the mouth could then be seen explicitly in the solution.

Baffin Bay, Davis Strait and the Labrador Sea, when schematized as rectangular seas of constant depth, happen to satisfy very well Grace's assumption of the smallness of $(\frac{\sigma\alpha}{gh})^2$ and this will lead to great simplifications.

General Summary

The diurnal tides and semidiurnal tides in the Labrador Sea, Davis Strait and Baffin Bay have been investigated with the help of the following assumptions:

1. - The sea under study is closed in its sides and northern boundaries.

2. - The wave motion is of a purely standing character with perfect reflection taking place at the head.

3. - The tide is induced exclusively by that in the Atlantic Ocean.

4. - The friction is negligible throughout the sea.

Initially we assume further that the motion is purely unidirectional. The equations are then very simple to solve and the form of the solution depends on the law of variation chosen for the width and depth. We chose an exponential law of width and depth when a schematization of constant depth and width was not obvious. By dividing the sea into an increasing number of channels it was possible to reproduce more and more closely the physical features of the sea. Fig. 4 illustrates the various steps in our schematization. It was then possible to obtain solutions for the elevation Z and the current v and the curves for Z are compared with the observed values. This is shown in Fig. 5 and Fig. 7. Even the more refined 6 channel model fails to locate the nodes accurately.

The assumption of perfect reflection at the northern boundary was then dropped and transports were assumed at this point while perfect reflection was assumed to take place at some undetermined point further beyond our chosen boundary for the sea. In this way it was possible to bring the node to its observed location. Fig. 6 and 8 illustrate the transports corresponding to the various Z curves; such transports imply estimates of the currents about which no information is available.

Once the position of the nodes and the transports at the head were established with the help of the one dimensional calculations, the problem of two dimensional motion was approached by schematizing the sea into three rectangular basins of constant depth. This took roughly into account the variation in width and depth while it was possible to obtain a solution of the equations of hydrodynamic which was not too difficult to handle. It was noticed that each basin satisfied very well the condition of narrowness and depth already noticed by Grace (3). The consequences of this were systematically investigated using as a powerful auxiliary the concept of linear vector space. In the body of the thesis we state and prove the following theorems:

In a narrow deep and elongated sea,

1. - The Poincaré waves are significant only in the vicinity of the boundaries

and are rapidly damped away from them.

A corollary is:
The motion tends to become unidirectional away from the boundaries.

2. - If motion is induced at one section, only the contribution to the Kelvin waves will propagate far into the sea, the contribution to the Poincaré waves will rapidly be damped.

We have two corollaries:

a. Only the part of the energy which is transmitted to the Kelvin wave will propagate into the sea.

b. Only the part of the current which is constant across the mouth will propagate into the sea.

3. - A Kelvin wave can always be reflected in a rotating sea of any length.

Once these theorems are established it is a simple matter to state and satisfy the boundary conditions. The calculations are lengthy and tedious but straightforward. We obtain a solution for M_2 which converges everywhere except at the inner corners while the solution for K_1 diverges at the inner boundaries. The results of such calculations are shown in Fig. 9 to 12.

Basic Approximations

We wish to assume first that the sea under study is approximately closed in its sides and in its northern boundary.

The main opening on the sides is Hudson Strait. The cross section at the mouth of Hudson Strait is 20 km^2 while the section across the sea just north of the opening of the strait (around 62° 30'N) is 630 km^2 . The ratio of the two cross sections is of 1 to 31.

Baffin Bay ends into the three sounds of Lancaster, Jones and Smith. We assume that the bay is closed at the section marked by the double lines shown in Fig. 1 around latitude 75° N. The largest section of Baffin Bay (around 72° 30'N) is 430 km^2 while those of Lancaster and Smith Sounds are 73 km^2 and 71 km^2 respectively. The section of Jones Sound is negligible. The assumption of complete closure at the northern end is therefore not too strong and will have to be relaxed eventually.

Friction is not taken into consideration. The maximum tidal streams over the body of the sea (except in the immediate vicinity of the head) are computed to be of the order of 15 cms/sec and the minimum mean depth is 300 m. In such a situation the effect of friction is of secondary importance.

The tides are considered as being exclusively induced by those in the Atlantic; the tides caused in situ by the tidal potential are neglected. The following calculations will give and idea of the magnitudes involved.

The force due to M_2 and K_1 is given by

$$F = g\ (\ .7607 x 10^{-7} .\ \cos^2 L,\ \ .4442 x 10^{-7}\ \ \sin 2L\ )$$

where L is the latitude and where the time varying part has been removed (see Schureman (13)).

The sea under consideration can be roughly considered as a rectangular basin of constant depth 1120m and of length 2350 km lying between latitude 55° and 75°N (.96 to 1.31 radians) at an angle to the meridians; we wont specify this angle but it will be such that L is approximately given by

$$L = 1.31 - fy$$

where y is the distance measured from the head and f has for value $1.49 x 10^{-4}$ km^{-1} so that L = .96 radians when y = 2350 kms.

Deriving an equation for Z from equations (1-1) and (1-2) in Appendix 1 with the added force terms we obtain

$$\frac{d^2 Z}{dy} + K^2 Z = (.7607 x 10^{-7}\ \ f \sin 2L,\ \ .8884 x 10^{-7}\ \ \cos 2L\ ) \text{ for}$$

M_2 and K_1

This is a non homogeneous linear differential equation with constant coefficients which has for solution

$$Z = A \cos Ky + B \sin Ky + \left(.7607\text{x}10^{-7} \frac{f \sin 2L}{k^2 - 4f^2},\ -.8884\text{x}10^{-7} \frac{f \cos 2L}{k^2 - 4f^2}\right)$$

or explicitely

$$Z = A \cos Ky + B \sin Ky + (.66 \sin 2L,\ -3.4 \cos 2L)$$

where A and B are arbitrary constants of integration. If we impose as boundary conditions $Z = 0$, $v = 0$ at $y = 0$, the solution becomes

$$Z = (-.32 \cos Ky - .13 \sin Ky + .66 \sin 2L,\ -2.97 \cos Ky -.37 \sin Ky - 3.4 \cos 2L)$$

The largest values of Z, which obtain at the mouth, are .93 cms and .79 cm for M_2 and K_1 respectively. This type of motion can then be safely neglected.

ONE DIMENSIONAL THEORY

The Equation of Motion in One Dimension for a Fluid and the Boundary Conditions:

The Solutions

The equations of motion for a fluid of density 1 in one dimension are:

$$\frac{\partial v}{\partial t} = - g \frac{\partial Z}{\partial y} \qquad (1)$$

$$\frac{1}{b} \frac{\partial}{\partial y} (b h v) = - \frac{\partial Z}{\partial t} \qquad (2)$$

These hold for a channel of rectangular section, width b, depth h. (1) states Newton's second law while (2) states the law of conservation of mass. The conditions of validity of these equations are well known and will not be reproduced here.

Z and v are periodic functions of time in the phenomena under study and we write, assuming standing waves,

$$Z(y,t) = Z(y) \quad \cos \sigma t$$

$$v(y,t) = v(Y) \quad \sin \sigma t$$

where σ represents the rate of change of the phase of the given constituent and t is the time.

This notation leads to no ambiguity since from now on we shall be concerned only with the space part of Z and v. The equations satisfied by each of these functions are now

$$v = - \frac{g}{\sigma} \frac{dZ}{dy} \qquad (3)$$

$$\frac{1}{b} \frac{d}{dy} (b h v) = \sigma Z \qquad (4)$$

Our models will consist of one or more channels open at one or both ends or closed at one end. We therefore wish to solve (3) and (4) for the following boundary conditions.

Z = 1 cm at the mouth

v = 0 cm/sec at the head.

We impose the boundary value unity on Z because (3) and (4) are linear equations and one needs simply to multiply the solutions obtained with such a boundary value by Z_o if this happens to be the value observed at the mouth.

Since our models may consist of more than one channel we can determine the arbitrary constants completely only if we impose further conditions relating the elevations and the transports at the junctions of every two channels. The standard connecting conditions holding at the junction of two channels, say 1 and 2 are

$$Z_1 = Z_2 \qquad (5)$$

$$b_1 h_1 v_1 = b_2 h_2 v_2 \qquad (6)$$

(5) excludes any discontinuity in amplitude while (6) even if it implies a possible jump in v conserves the mass of liquid flowing per unit time from one channel to another.

The solution of (3) and (4) depends on the functional dependence of b and h on y.

The solution comes out in terms of

1. - Trigonometric functions, if b and h are constant,

2. - Bessel functions, if b and h vary exponentially,

3. - Confluent hypergeometric functions, if b varies exponentially while h varies linearly,

4. - Hypergeometric functions, if b is constant while the depth is described by a parabola,

5. - Infinite series not related to any standard functions, when b and h vary linearly (These series represent Bessel functions of order 0, 1 and 2 when the width is constant or when the depth and width vary at the same rate).

Preliminary calculations have been done using these various laws when they were approximately equivalent to describe parts of the b h curve given in Fig. 4; the results have been found to be almost identical. In the final calculations presented here an exponential law of width and depth has been used throughout except when a schematization of constant width and depth was obvious. The general solution of (3) and (4) for an exponential law of width and depth comes out in terms of Bessel functions of the first kind and usually of a non integral order; it is derived in Appendix 1.

Although these functions are not tabulated and have to be evaluated from their series expansions, the convergence is rapid, which is not the case for the functions corresponding to the other laws of width and depth. More, extent tables of Bessel functions of an integral order helped to check the calculations done using the Bessel functions of non integral order while it was more difficult to check the other calculations using the solutions involving the more esoteric functions which are only covered by rudimentary tables.

To set up the coastal observations on amplitude on a linear scale we drew a line in the middle of the sea following the trend of the coast. Distance was measured along this line and divisions approximately perpendicular to it were drawn to reach the coasts. In this way the amplitudes were plotted as in Fig. 5 or 7, the full points denoting the observations on the East Coast while the open circles give those on the West Coast. Points considered to lie beyond the node were plotted on the negative axis.

The non schematized plot of the width and depth variation along the sea shown for instance in Fig. 4 gives a three dimensional picture of the basin. One imagines the b plot to be reflected across the y axis (and reduced by one half) and this gives a top view of the basin, while the h plot indicates the variation of depth along it.

Table 3 that follows give the data on the schematization of this basin into 1, 2, 3 and 6 channels while Fig. 4 illustrates this schematization except in the one channel case.

The schematization into two channels represents roughly the Labrador Sea connecting into Baffin Bay, Davis Strait being done away with altogether; the minimum of depth and width are put together which is not the case in the actual sea. The three channel model includes Davis Strait while the six channel model molds itself reasonably well to all parts of the basin.

Figs. 5 to 8 show the results of the calculations carried through for M_2 and K_1. Appendix 2 gives the corresponding analytic solutions. These solutions hold for the boundary value $Z_0=1$ at the mouth. To obtain the curves shown in Figs. 5 and 7 the Z solution has been multiplied by 70 and 14 respectively for M_2 and K_1 since these seem reasonable boundary values; the transports however correspond to the normalized solutions. The latter part of Appendix 2 gives the currents corresponding to the six channel model since in that instance they are continuous.

Table III

DATA ON THE SCHEMATIZATION INTO CHANNELS

NO OF CHANNELS		LENGTH KM	WIDTH KM	βxl	DEPTH M	αxl	β/α
1		2350			1120		
2	1	1140	970→300	-1.17351	2600→310	-2.12346	.5526
	2	1210	450		800		
3	1	1000	970→420	.83725	2600→310	2.12704	.3936
	2	140	420→300	-.33647	310→400	.24822	-1.3555
	3	1210	450		800		
6	1	600	970→800	-.19268	2600→1600	-.48551	.3969
	2	400	800→420	-.64434	1600→310	-1.63797	.3934
	3	140	420→300	-.33647	310→400	.24822	-1.3555
	4	260	300→480	.47000	400→885	.79412	.5918
	5	800	480		885		
	6	150	480→290	-.50410	885→250	-1.26413	.3987

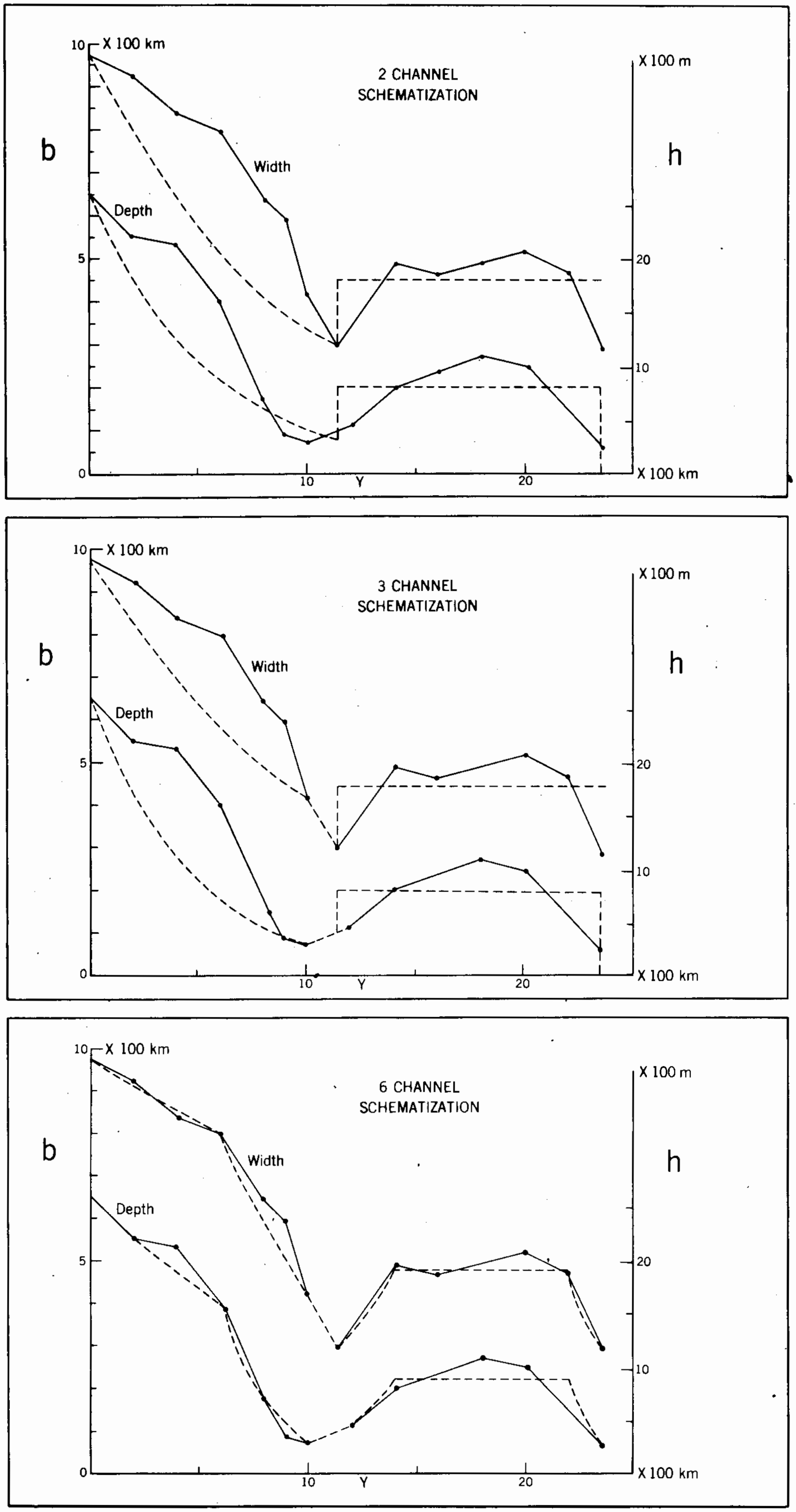
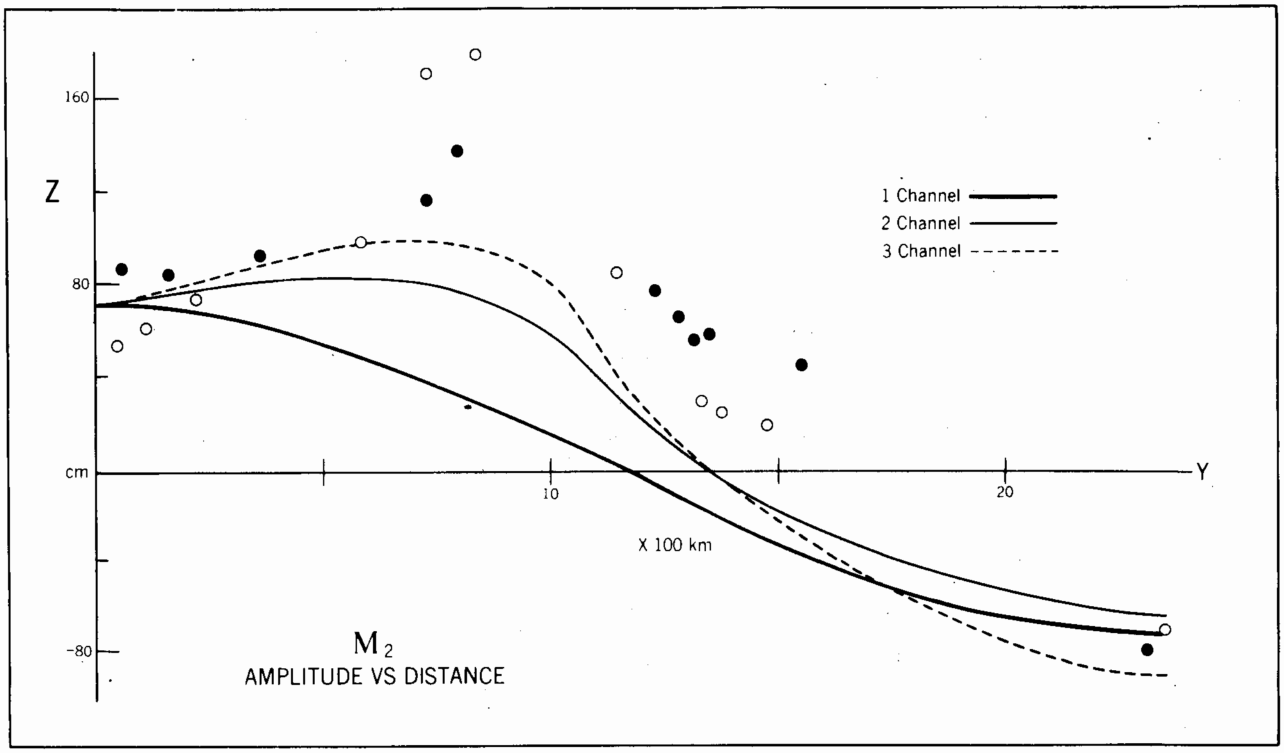

10
X 100 km
X 100 m
2 CHANNEL SCHEMATIZATION
b
h
Width
Depth
5
20
10
0
10
Y
20
X 100 km
10
X 100 km
X 100 m
3 CHANNEL SCHEMATIZATION
b
h
Width
Depth
5
20
10
0
10
Y
20
X 100 km
10
X 100 km
X 100 m
6 CHANNEL SCHEMATIZATION
b
h
Width
Depth
5
20
10
0
10
Y
20
X 100 km

Fig. 4


160
Z
80
cm
-80
10
20
Y
X 100 km
1 Channel
2 Channel
3 Channel
M2
AMPLITUDE VS DISTANCE

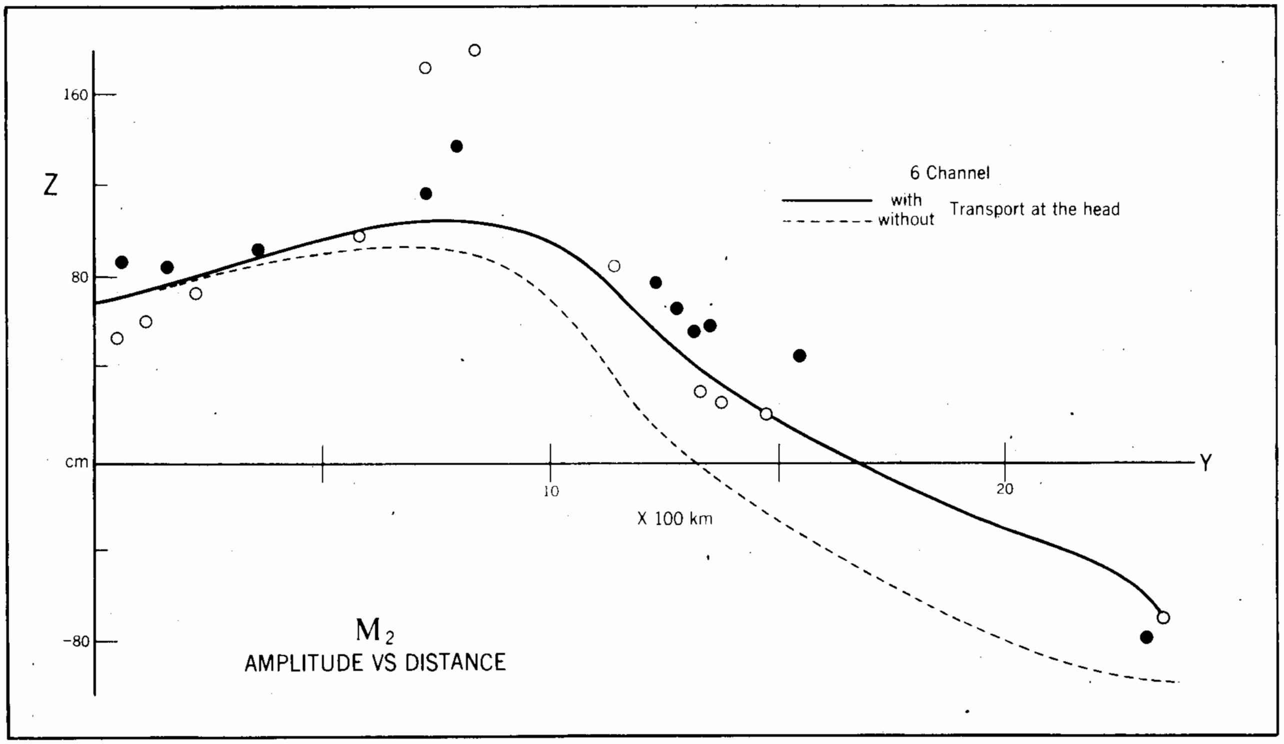

160
Z
80
cm
-80
10
20
Y
X 100 km
6 Channel
with
without
Transport at the head
M2
AMPLITUDE VS DISTANCE

Fig. 5

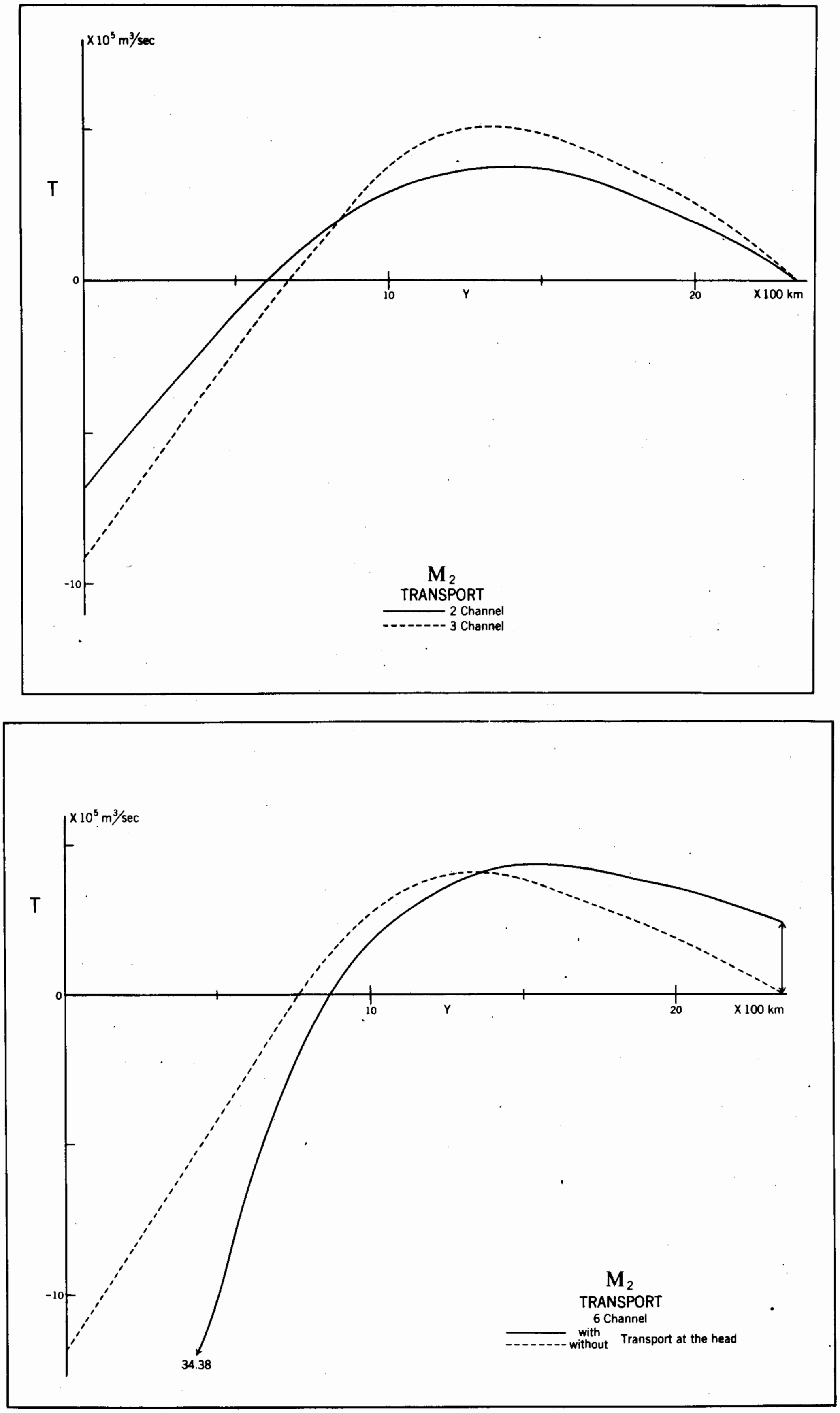
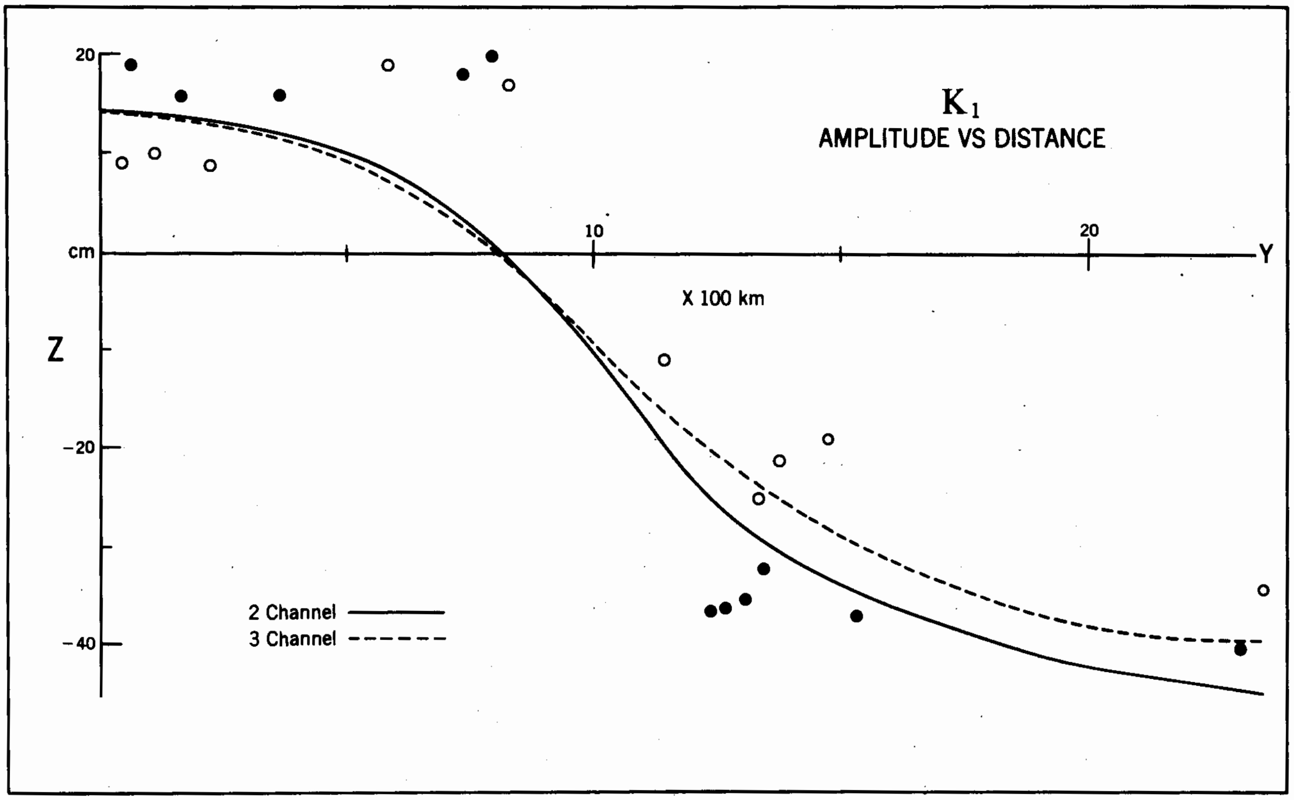

X 10^5 m^3/sec
T
0
-10
10
Y
20
X 100 km
M$_2$
TRANSPORT
2 Channel
3 Channel
X 10^5 m^3/sec
T
0
-10
10
Y
20
X 100 km
34.38
M$_2$
TRANSPORT
6 Channel
with
without
Transport at the head

Fig. 6


K₁
AMPLITUDE VS DISTANCE
20
cm
Z
-20
-40
10
20
Y
X 100 km
2 Channel
3 Channel

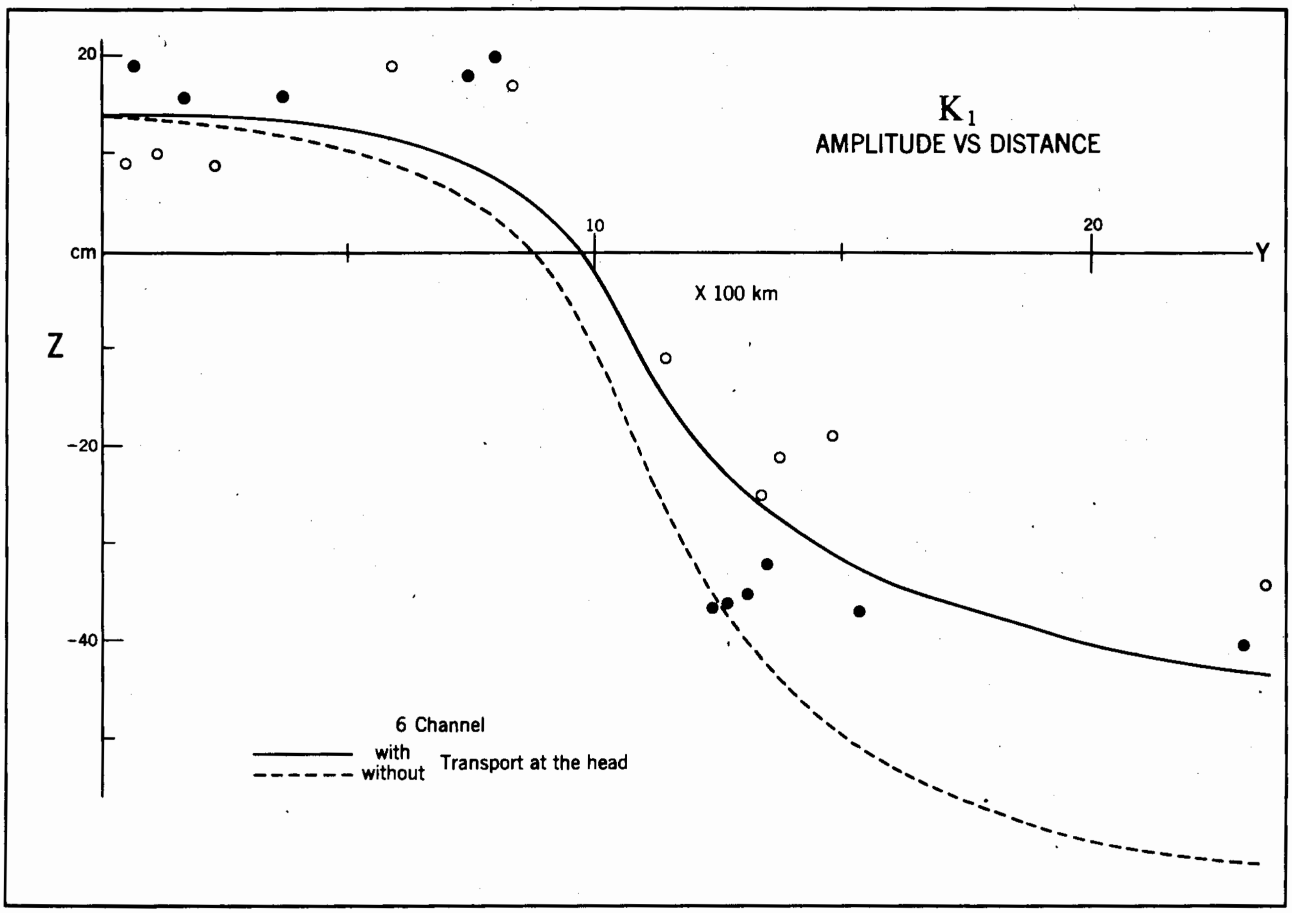

K₁
AMPLITUDE VS DISTANCE
20
cm
Z
-20
-40
10
20
Y
X 100 km
6 Channel
with
without
Transport at the head

Fig. 7

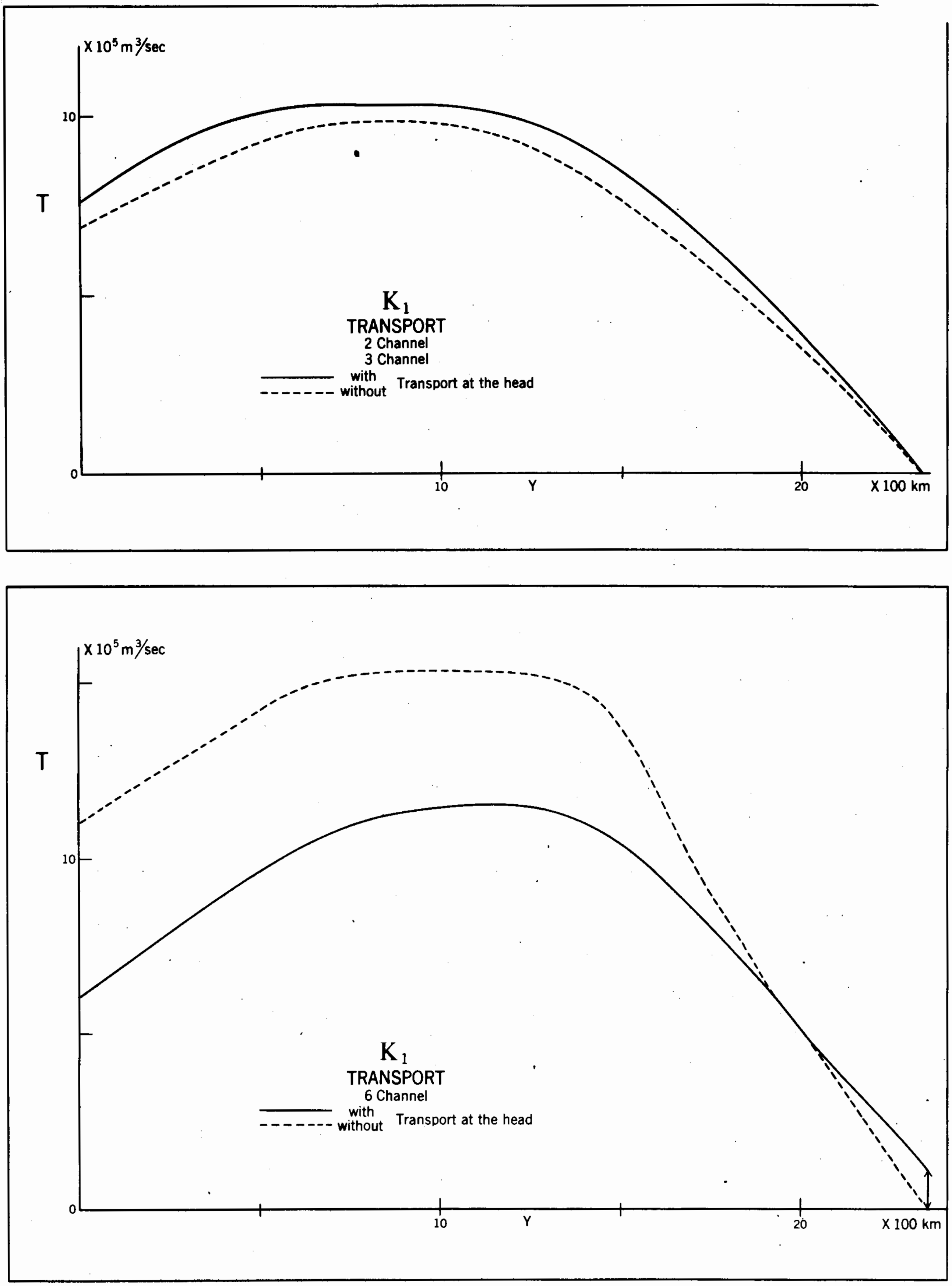

X 10⁵ m³/sec
10
T
0
K₁
TRANSPORT
2 Channel
3 Channel
with
without
Transport at the head
10
Y
20
X 100 km
X 10⁵ m³/sec
10
T
0
K₁
TRANSPORT
6 Channel
with
without
Transport at the head
10
Y
20
X 100 km

Fig. 8

Discussion of the Results

In Fig. 5 the 1 channel approximation for M_2 although very rough indicates the presence of a node half way up the channel; it fails however to show any amplification of the amplitude and the node is too close to the mouth. The 2 and 3 channel models show some amplification but still fail to bring the node forward enough. The 6 channel model, assuming perfect reflection gives quite similar results. Assuming a transport of 1.75×10^7 m^3/sec (for Z_o = 70 cm) at the head brings the node close to where it most likely lies. Some increase in amplification and a forward shift in the maximum is noted but such an amplification is far short of that which actually occurs on the west side of the sea. The amplification on the Greenland side is fully accounted for since the calculated amplification is of the order of 150% which is that observed on the Greenland coast. At the head of the sea the curve shows some steepening which indicates that there is little change in amplitude in the northern part of Baffin Bay till one reaches Lancaster Sound and Smith Sound. There is no data to support or contradict this deduction. The currents at the head are computed to be of the order of 24 cm/sec for a Z_o value of 70 cm.

In Fig. 7 the one channel approximation for K_1 is not shown. In this approximation, the node lies just inside the sea; this would indicate resonance, which is not the case. The 2 and 3 channel models give quite a reasonable fit to the amplitude assuming perfect reflection. The 6 channel model using the same assumption apparently leads to worse results. Assuming a slight outflow at the head of 1.7×10^6 m^3/sec, being equivalent to a stream of 2 cm/sec across the section, brings the curve closer to the observed values. This outflow corresponds to a boundary value Z_o of 14 cm.

If one inspects Table 1, one notes that in Lancaster Sound the amplitude of M_2 and K_1 at Dundas Harbour (Station 5) is larger than that at Radstock Bay (Station 6). There must be a maximum in the amplitude either between stations 5 and 6 or ahead of Dundas Harbour. Assuming this last instance to be the case and the wave in this area to be of a perfect standing character, there should be weak streams in the vicinity of Dundas Harbour and the assumption of the closure of the sea in this area should hold well.

When we come to Smith Sound the situation is quite different. Thule (Station 28) shows the largest amplitude in K_1 and the diurnal streams should be weak in this area thus allowing us to assume perfect reflection of K_1 at this boundary. But the maximum in the amplitude of M_2 is to be found in the neighbourhood of Port Foulke (Station 27) and Rensselaer (Station 26) further up in Smith Sound. There are large gradients in the amplitude, from 66 cm at Thule to 111 cm at Port Foulke and therefore appreciable semidiurnal currents should be expected to prevail at the chosen northern boundary of the sea.

One notices the large increase in transport at the mouth for M_2 for a slight increase in outflow at the head. Mathematically this can be explained by the steepening of the gradient at the mouth for increasing outflows. The currents at the mouth are larger and these flowing across a very large section will cause much increased transports. On the other hand such large transports are physically possible at the mouth of the sea since there exist very large gradients in the amplitude of M_2 along the Labrador Coast and even through these gradients do not extend all the way across the sea to Greenland, they must extend up to a certain distance from the west coast and cause very marked transports.

Comparison with Defant's Calculations

Defant (2) investigated the M_2 tide in the area covered by this thesis.

His data on width and depth are almost identical to those presented in Fig. 4 for the Labrador Sea and Davis Strait. However the depths shown for Baffin Bay are consistently lower than those that can be abstracted from contemporary bathymetric charts.

Using a numerical scheme for the integration of equations (3) and (4), M_2 was calculated for a 10 channel model assuming complete reflection at the head. The results indicate a node at $y = 1550$ km from the mouth (our calculations with the same assumption would locate it at 1350 km and a maximum in amplitude at about $y = 830$ km); this corresponds very closely to reality.

The success of these rather crude calculations using a defective model is surprising at first. However it must be realized that shallower depths in Baffin Bay automatically bring the node forward towards the head.

It was decided to repeat Defant's calculations using his data on width and depth; first for a 20 channel model of the sea using his numerical scheme, then for a 3 channel model consistent with our approach and defined by the following quantities

$l_1 = 950$ km $\quad b_1 = 969 \rightarrow 365$ km $\quad h_1 = 2610 \rightarrow 350$ m

$l_2 = 240$ km $\quad b_2 = 365$ km $\quad h_2 = 350$ m

$l_3 = 1190$ km $\quad b_3 = 471$ km $\quad h_3 = 610$ m

The results are given in Table 4. Displacements rather than currents are shown. The amplitudes are normalized to 100 at the mouth and such data on amplitude that were derived from the numerical schemes were interpolated to yield values at the mouth and head of the channels; the ξ's for the 3 channel model are omitted since they are discontinuous.

It can be noticed that the finer 20 channel scheme which should give a solution closer to the exact solution of (3) and (4) for the sea described by Defant modifies his solutions in two ways:

1) The maximum is reduced from 151 to 138,
2) The node is pushed away from the head and lies at about 1450 (the maximum as well being brought to lie at 700 km).

The 3 channel model although itself quite coarse tends to agree with the 20 channel model.

Table IV

Defant's Calculations Repeated

	AMPLITUDE			DISPLACEMENT	
	10 CHANNEL	20 CHANNEL	3 CHANNEL	10 CHANNEL	20 CHANNEL
CHANNEL JUNCTION	Z CM	Z CM	Z CM	ξ M	ξ M
0	100	100	100	387	371
0. 5		109			371
1	119	117	115	374	362
1. 5		124			285
2	136	130	125	298	238
2. 5		134			215
3	151	138	127	257	159
3. 5		136			024
4	151	124	120	- 260	- 471
4. 5		97			- 1067
5	95	60	56	- 1910	- 1911
5. 5		26			- 1364
6	20	0	15	- 1043	- 1038
6. 5		-23	0		- 963
7	-30	-43	-28	- 878	- 838
7. 5		-59			- 694
8	-66	-72	-63	- 558	- 533
8. 5		-81			- 376
9	-86	-87	-88	- 224	- 216
9. 5	(-92)	-90			- 156
10			-96	0	0

Conclusions

The one dimensional theory confirms the presence of nodes in the amplitudes of M_2 and K_1 in the area under study. It fails however to locate these nodes accurately unless some outflow is assumed at the head of the sea; these outflows correspond to currents of 24 cm/sec and 2 cm/sec for M_2 and K_1 respectively.

It supports rather well the variation in the amplitude of M_2 and K_1 along the sea; large amplitudes of the diurnal wave are calculated in Baffin Bay in spite of its smallness at the junction with the Atlantic and some amplification of M_2 ahead of Davis Strait is calculated.

The theory also has the advantage of supplying estimates of the currents prevailing in the sea and about which no information is available. It gives a mean current of 10 cm/sec for M_2 at the mouth of the basin and it suggests that the diurnal currents are small over most of the sea except in Davis Strait where they reach a magnitude of 13 cm/sec.

The one dimensional theory obviously cannot explain why the point of amphidromy for K_1 is degenerate, it cannot take account of the large gradients in the amplitude of both constituents across the mouth of the sea nor of the large difference in gradient along the east and west coast. It does not give either any ides of the variation in width of the currents across the channels and of the transverse currents.

All this is the task of the two dimensional theory.

TWO DIMENSIONAL THEORY

Summary

We will study the two dimensional motion in a sea made up of three rectangular basins of constant depth, one basin corresponding to the Labrador Sea, Davis Strait and Baffin Bay respectively. Such a drastic simplification of the features of the domain under study is necessary in order to avoid excessive analytic complication and even in this very simple model one is faced with elaborate calculations.

We shall first derive the solution of the set of three partial differential equations which describe u, v, and Z in a rotating rectangular sea. Then we shall simplify this solution with the help of some approximations which hold well in the three seas under study; a sea satisfying these simplifying assumptions will be called narrow, deep and elongated (from now on denoted as a nde sea).

Before attempting to state and satisfy the boundary conditions that the solution has to satisfy in the model, we will introduce the concept of linear space. With the help of this concept we shall express the solution in terms of basic vectors which describe it fully in various spaces.

Once the solution is expressed in terms of the basic vectors it will be possible to understand more clearly its behaviour over the body of the sea and in the immediate vicinity of the boundaries.

We shall elucidate some statements by Taylor about the reflection of Kelvin waves in a rotating rectangular sea and we shall show that in a sea of finite dimensions and which is not too shallow, perfect reflection is always possible.

We will see as well that in a nde sea, only the part of the current which is constant across the mouth will be transmitted to the next boundary; the part of the current which is a function of x will only affect the first sea and will not reach the next one.

Once this is established we shall be in a position to state and satisfy the boundary conditions that the solution has to satisfy over our three basin model. In vector language this corresponds to equating two vectors which in turn implies that each of their components should be equal. This leads to a set of linear equations relating the undetermined constants contained in the solution thus allowing in principle a complete resolution of the problem.

With the help of the approximation of narrowness, depth and elongation we shall succeed in separating these linear equations into quasi independent subsets and eliminate most unknowns. Eventually we shall be faced with the task of inverting two 20 x 20 matrices.

Charts showing the elevation and currents will be drawn from the result of such calculations.

The Model

We use two different models for M_2 and K_1 ; the reasons for such a procedure are given below. The models are represented by the following quantities:

M_2

$l_1 = 1000$ Km $\quad b_1 = 762$ Km $\quad h_1 = 1570$ M

$l_2 = 200$ Km $\quad b_2 = 354$ Km $\quad h_2 = 370$ M

$l_2 = 1450$ Km $\quad b_3 = 439$ Km $\quad h_3 = 800$ M

K_1

$l_1 = 1000$ Km $\quad b_1 = 762$ Km $\quad h_1 = 1570$ M

$l_2 = 200$ Km $\quad b_2 = 177$ Km $\quad h_2 = 370$ M

$l_3 = 1150$ Km $\quad b_3 = 439$ Km $\quad h_3 = 800$ M

All the b and h quoted are averages of the observed b and h with the exception of b_2 for K_1.

In order to test the quality of the model we have subjected it to one dimensional computations for three channels of constant depth and width. Since it was important to preserve the position of the nodes where observation puts them, we had to make sea 3 (Baffin Bay) longer by 300 Km for M_2 in order to bring the node to 1660 Km. Within this model a transport of 2.6×10^5 m^3/sec is computed to take place at 2350 Km which compares favourably with the value of 2.5×10^5 m^3/sec assumed in the one dimensional theory (For $Z_o = 1$ Cm).

For K_1 the position of the node depends very sensitively on the variation of width in Davis Strait and the only way to bring the node to 1040 km with the above model is to cut b_2 by one half.

In both models perfect reflection is assumed to take place at the head.

An illustration of the model is given on the next page.

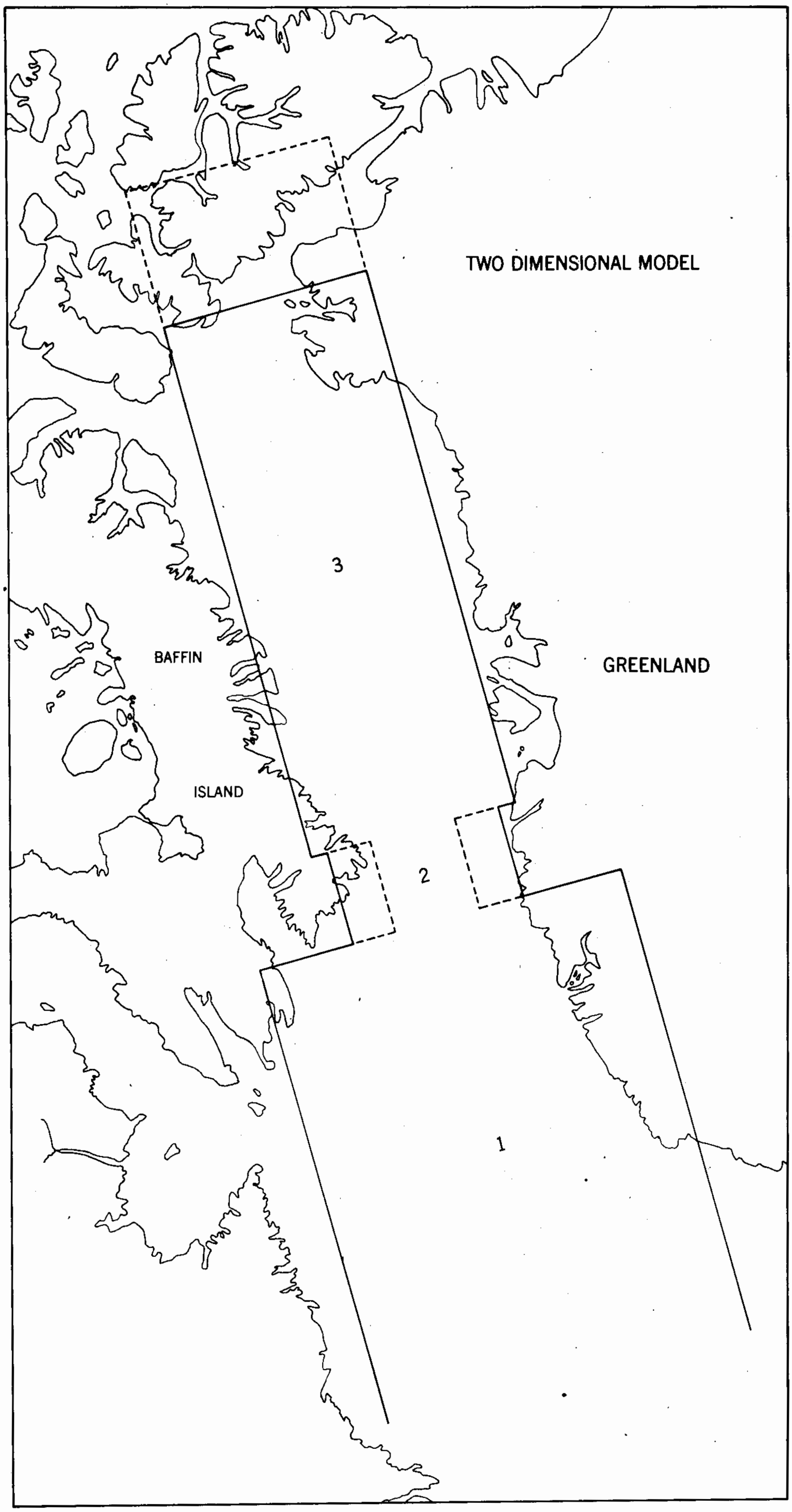


Fig. 9

The Solution of the Equations of Hydrodynamics for a Rotating Rectangular Sea of Constant Depth

The equations of hydrodynamics in two dimensions in a Cartesian frame of reference rotating at an angular speed ω are (Lamb (7)).

$$\frac{\partial u}{\partial t} - 2\omega v = -g\frac{\partial Z}{\partial x} \qquad (7)$$

$$\frac{\partial v}{\partial t} + 2\omega u = -g\frac{\partial Z}{\partial y} \qquad (8)$$

$$\frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = -\frac{\partial Z}{\partial t} \qquad (9)$$

(7) and (8) express Newton's second law in the x and y direction and (9) states the conservation of mass. No external forces are acting on the fluid and the density is taken to be equal to 1.

The time variation of the variables u, v and Z is written as $e^{i\sigma t}$ where i is the imaginary unit. The real part of the variable is its value at time 0; its imaginary part, its value 1/4 period after. All the numbers we will handle from now on will be complex.

If in the basin under study, the depth is constant, (7), (8) and (9) take the form

$$i\sigma u - 2\omega v = -g\frac{\partial Z}{\partial x} \qquad (10)$$

$$i\sigma v + 2\omega u = -g\frac{\partial Z}{\partial y} \qquad (11)$$

$$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\sigma Z = 0 \qquad (12)$$

This is a set of simultaneous partial differential equations for the unknowns u, v and Z, and the basin over which they prevail is described by the equations

$$x = \pm a \qquad y = \pm L$$

The basin is therefore rectangular, of width 2a and length 2L with the origin being taken at its center. The basin is always closed on its sides i. e. at $x = \pm a$, but it may be open or closed at its extremities.

We need out of the general solution of (10) to (12) the special solution satisfying the boundary condition

$$u(\pm a) = 0 \qquad (14)$$

The set (10) to (12) admits of a solution satisfying

$$u = 0 \qquad (15)$$

everywhere, therefore satisfying (14) a fortiori. (15) leads to three relations joining v and Z; these relations are consistent and therefore yield the solution

$$u = 0 \qquad (16)$$

$$v = A_o \sinh[\lambda' x - iK'_o y] + B_o \cosh[\lambda' x - iK'_o y] \qquad (17)$$

$$Z = \left(\frac{h}{g}\right)^{\frac{1}{2}} \{ A_o \cosh[\lambda' x - iK'_o y] + B_o \sinh[\lambda' x - iK'_o y] \} \qquad (18)$$

$$\text{where } \lambda' \equiv \frac{2\omega}{(gh)^{\frac{1}{2}}} \text{ and } K'_o \equiv \frac{\sigma}{(gh)^{\frac{1}{2}}}$$

Ao, Bo are arbitrary constants.

This is the Kelvin solution and it would have been lost had we tried to obtain independent partial differential equations for each variable. Such a solution is not the only one satisfying equations (10) to (12) and the boundary condition (14); there exist a full infinity of extra solutions. But we shall soon see that the Kelvin solution describes the preponderant part of the motion in a sea of the dimensions described in the previous paragraphs.

To obtain the remaining solutions we eliminate v and Z; the equation satisfied by u is

$$(\nabla^2 + k^2)\, u = 0 \qquad (19)$$

This is Helmholtz equation where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$k^2 \equiv \frac{\sigma^2 - 4\,\omega^2}{g\,h}$$

v and Z satisfy (19) as well but we are not concerned with that. It is convenient at this point to write the relations linking u and v to Z, which will become useful in subsequent work.

$$u = \frac{g}{\sigma^2 - 4\omega^2}\left\{2\omega\,\frac{\partial Z}{\partial y} + i\,\sigma\,\frac{\partial Z}{\partial x}\right\} \qquad (20)$$

$$v = \frac{g}{\sigma^2 - 4\omega^2}\left\{-2\omega\,\frac{\partial Z}{\partial x} + i\,\sigma\,\frac{\partial Z}{\partial y}\right\} \qquad (21)$$

We wish to obtain solutions of (19) which satisfy condition (14). (19) is separable and its solution can be written in the form

$$X(x)\,Y(y)$$

so that (19) becomes

$$\frac{1}{X}\,\frac{d^2X}{dx^2} + \frac{1}{Y}\,\frac{d^2Y}{dy^2} + k^2 = 0 \qquad (22)$$

We may write $\quad \dfrac{1}{X}\,\dfrac{d^2X}{dx^2} = -\,j'^2 \qquad (23)$

where $\quad j' \equiv \dfrac{j\,\pi}{2\,a}$

and j is as yet an undetermined constant.

(23) implies

$$\frac{1}{Y}\,\frac{d^2Y}{dy^2} = j'^2 - k^2 \qquad (24)$$

(23) and (24) have solutions of the form

$$X \sim e^{\pm ij'x}$$

$$Y \sim e^{\pm K_j (\frac{\pi y}{2a})}$$

where

$$K_j \equiv \left\{ j^2 - k^2 \cdot \left(\frac{2a}{\pi} \right)^2 \right\}^{\frac{1}{2}}$$

The notation is somewhat simplified if we redefine the y variable as

$$\frac{\pi y}{2a} \rightarrow y$$

The interval over which it is defined is $-\frac{\pi L}{2a} \leqslant y \leqslant \frac{\pi L}{2a}$

and we set

$$\frac{\pi L}{2a} \equiv 1$$

a dimensionless number.

Then the y solution takes the form

$$Y \sim e^{\pm K_j y}$$

It is not convenient to renormalize the x coordinate in the same way; this will eventually become obvious.

The solution for u has the form

$$u \sim e^{\pm K_j y} e^{\pm ij'x}$$

By suitable linear combinations of such solutions we get expressions of the form

$$u \sim e^{\pm K_j y} \begin{Bmatrix} \sin j'x \\ \cos j'x \end{Bmatrix}$$

u will satisfy (14) if

j = an even integer for the sine solution

j = an odd integer for the cosine solution

The general solution of (19) is then

$$u = \sum_{\substack{m=1 \\ \text{odd}}}^{\infty} \left(C'_m e^{K_m(y-l)} + D'_m e^{-K_m(y+l)} \right) \cos m'x \qquad (25)$$

$$+ \sum_{\substack{n=2 \\ \text{even}}}^{\infty} \left(A'_n e^{K_n(y-l)} + B'_n e^{-K_n(y+l)} \right) \sin n'x \; .$$

The primed arbitrary constants are related to the unprimed constants by the relations

$$C'_m \equiv \left\{ \frac{4\omega^2 + ghm'^2}{ghm K_m} \right\} \cdot \left(\frac{2a}{\pi} \right)^2 \cdot C_m \qquad \text{etc.}$$

From now on $\underline{m}$ will always denote an odd integer, $\underline{n}$ an even integer and $\underline{j}$ any integer.

A_n, B_n, C_m, D_m, form a two fold infinity of arbitrary constants and can be written out as

$$C_1, A_2, C_3, A_4, \ldots.$$

$$D_1, B_2, D_3, B_4, \ldots.$$

We have introduced the exponent in $K_j l$ in order to eliminate any exponential factors from the linear relations linking the arbitrary constants that will arise when further boundary conditions will be imposed. The solutions corresponding to an odd integer are of an even character and those corresponding to an even integer are of an odd character. In future we will often refer to a solution affected by an arbitrary constant as the wave pertaining to that constant, thus:

"The A_0 Kelvin wave, the C_m wave etc."

Out of (25) we can derive the corresponding solutions for v and Z using relations (10) to (12). These are, including the Kelvin solution:

$$v = A_o \sinh[\lambda' x - iK_o(y-1)] + B_o \cosh[\lambda' x - iK_o(y-1)]$$

$$+ \sum_{m=1}^{\infty} \left(C_m e^{K_m(y-1)} - D_m e^{-K_m(y+1)} \right) \sin m' x$$

$$- \sum_{n=2}^{\infty} \left(A_n e^{K_n(y-1)} - B_n e^{-K_n(y+1)} \right) \cos n' x$$

$$+ i\,\Omega \left\{ \sum_{m=1}^{\infty} \left(C_m e^{K_m(y-1)} + D_m e^{-K_m(y+1)} \right) \frac{\cos m' x}{mK_m} + \sum_{n=2}^{\infty} \left(A_n e^{K_n(y-1)} + B_n e^{-K_n(y+1)} \right) \frac{\sin n' x}{nK_n} \right\} \tag{26}$$

$$Z = \left(\frac{h}{g} \right)^{\frac{1}{2}} \{ A_o \cosh[\lambda' x - iK_o(y-1)] + B_o \sinh[\lambda' x - iK_o(y-1)] \}$$

$$-2\omega\propto \left\{ \sum_{m=1}^{\infty} \frac{\cos m' x}{m} \left(C_m e^{K_m(y-1)} - D_m e^{-K_m(y+1)} \right) + \sum_{n=2}^{\infty} \frac{\sin n' x}{n} \left(A_n e^{K_n(y-1)} - B_n e^{-K_n(y+1)} \right) \right\}$$

$$- i\,\sigma\propto \left\{ \sum_{m=1}^{\infty} \frac{\sin m' x}{K_m} \left(C_m e^{K_m(y-1)} + D_m e^{-K_m(y+1)} \right) - \sum_{n=2}^{\infty} \frac{\cos n' x}{K_n} \left(A_n e^{K_n(y-1)} + B_n e^{-K_n(y+1)} \right) \right\} \tag{27}$$

$$\Omega \equiv \frac{2\omega\sigma}{gh} \cdot \frac{4a^2}{\pi^2} \qquad \propto \equiv \frac{2a}{\pi g} \qquad K_o \equiv \frac{2a}{\pi} K_o'$$

(25) to (27) represent a rigorous and complete solution of the set of partial differential equations (10) to (12) satisfying the boundary condition (14). The two-fold infinity of arbitrary constants they contain will be determined by applying extra boundary conditions at $y = \pm 1$.

The solution of Helmholtz equation satisfying boundary condition (14) exists only if the arbitrary constant j we introduced is an integer. Such values of j are eigenvalues and the solutions for u corresponding to $j = 1, 2, 3, \ldots,$ are eigenfunctions. The solutions for u, v and Z corresponding to these integral values are usually called the Poincaré waves.

The Kelvin solution can be considered as the u eigenfunction corresponding to $j = 0$. It is the equivalent of the solution we obtained in Appendix 1 for a channel of constant width and depth; this can be seen by letting $\lambda' \rightarrow 0$. The $e^{\pm\lambda' x}$ factor brings in the two dimensional features which are the sloping of the surface across the direction of motion and the variation in v necessary to balance this elevation.

The Poincaré waves exist only in two dimensions and would vanish completely in one dimensional motion.

Two Dimensional Motion in a Narrow Deep Elongated Sea of Constant Depth

Although (25) to (27) represent the motion in an exceedingly simple basin they are nevertheless quite complicated and not easy to manipulate. We will now use the features of the three basins under study, their narrowness, depth and length to simplify the v and Z solutions.

Algebraically a nde sea is characterized by the property

$$\frac{\sigma^2}{gh} \cdot \left(\frac{2a}{\pi}\right)^2 \ll 1 \qquad (28)$$

which is equivalent to the statement that half the wave length Λ is large compared to the width

$$\left[\frac{2a}{\Lambda/2}\right]^2 \ll 1$$

For seas 1, 2 and 3, (28) has the value

Sea	M_2	K_1
1	.076	.020
2	.070	.005
3	.049	.013

Criterion (28) is therefore satisfied for the three seas.

(28) implies

1) $K_j \sim j$ for the sea under study.

By definition

$$K_j \equiv \left\{ j^2 - \frac{\sigma^2 - 4\omega^2}{gh} \cdot \left(\frac{2a}{\pi}\right)^2 \right\}^{\frac{1}{2}}$$

The domain under study extends from latitude 55°N to 75°N.
for M_2 $\sigma \sim 2\omega$ while for K_1, $\sigma \simeq \frac{1}{3}$ to $\frac{1}{4}$ of 2ω.

So that for all three basins

$$\frac{\sigma^2-4\omega^2}{gh}\left(\frac{2a}{\pi}\right)^2 \sim 0 \qquad \text{for } M_2$$

$$\frac{\sigma^2-4\omega^2}{gh}\left(\frac{2a}{\pi}\right)^2 \sim -.01 \qquad \text{or less for } K_1$$

But we will still denote K_j by its own symbol in the work that will follow in order to keep track of this quantity.

2) $\Omega << 1$

Referring to the definition of Ω the above statement is obvious since $2\omega \sim \sigma$ for both constituents. Explicit calculations give

	Ω	
Sea	M_2	K_1
1	.07	.03
2	.06	.01
3	.04	.02

We shall as a consequence neglect Ω over the whole body of the sea. This means that we may drop the latter part of the solution for v; this is a very important simplification since it removes the coupling that would appear between the (A_n, B_n) and (C_m, D_m) constants when we try to satisfy the v boundary conditions. This approximation will lead to distinct sets of linear equations for the unknowns A_n, C_m, B_n, D_m.

No such simplification is possible for Z; all the Poincaré waves are of the same order of magnitude and the expression for Z has to be carried with all its complexity.

It will be convenient in two dimensions to impose boundary conditions on v and avoid manipulating Z as much as it is possible.

Now we wish to use the elongation of the sea in order to simplify our work further.

The Poincaré waves are all affected by exponents of the form

$$e^{\pm K_j(y\mp 1)} \sim e^{\pm j(y\mp 1)}$$

so that at y = ± 1 we will have terms of the form

$$\alpha_j e^{-2jl} \pm b_j \quad , \quad \alpha_j \pm b_j e^{-2jl}$$

At both boundaries one of the arbitrary constants is affected by an exponent

$$e^{-2jl}$$

If jl is large enough (the sea is elongated) the constant affected by this exponent may be neglected and in this fashion the Poincaré waves are separated into two distinct groups: (B_n, D_m) prevailing at the lower boundary and (A_n, C_m) at the upper boundary. The A_0 and B_0 waves are not affected by this factor and will be present at both boundaries; they will transmit the motion from one set of Poincaré waves to the other.

Let us now verify that the three basins are elongated:

e^{-2jl}				
Sea	j = 1		j = 2	
	M_2	K_1	M_2	K_1
1	.016	.016	.000	.000
2	.170	.029	.029	.001
3	.000	.003	.000	.000

This assumption is not justified for M_2, j = 1 in Sea 2. We will take care of this in our calculations.

The above shows that a Poincaré wave for which the exponential factor has value 1 at the boundary will be affected by a factor which will damp it rapidly away from the boundary. The Kelvin waves are unaffected by such an exponent and their amplitude is modulated trigonometrically throughout the length of the sea. Therefore irrespective of the relative magnitude of the Kelvin and Poincaré waves at the boundary, the Kelvin wave only will eventually survive as one moves up the sea. As a consequence, beyond a certain point, the cross current u becomes negligible and v and Z are determined almost exclusively by the amplitude of the Kelvin wave. In order to re-create more Poincaré waves one needs to introduce further boundaries into the path of the Kelvin wave: a wall, a constriction or a change in depth. At such a point the presence of Poincaré waves is essential to balance the Kelvin waves. The effect of these Poincaré waves will be felt ahead of the boundary but again not very far away from it.

In the case of co-oscillation, when the energy is introduced at the mouth of the sea, this energy will be distributed to the Kelvin and Poincaré waves; indeed it is necessary as a rule to have both kinds of waves in order to represent the motion at the mouth. However only the Kelvin wave will move unimpeded up the sea and therefore the portion of the energy which has been assigned to it will travel with it.

We summarize these remarks in two theorems:

Theorem 1. In a nde sea, the Poincaré waves are significant only in the vicinity of the boundaries.

Corollary. The current tends to become unidirectional away from the boundaries.

Theorem 1 and its Corollary may be restated in more poetic language as: *In a nde the Poincaré waves tune themselves in only in the vicinity of the boundaries while the Kelvin waves control the motion over the body of the sea.*

Theorem 2. *In a nde sea, in the case of co-oscillation, only the contribution to the Kelvin wave will propagate far into the sea.*

Corollary. *Only the part of the energy which is transmitted to the Kelvin wave will propagate into the sea.*

We shall restate Theorem 2 and its corollary in a much stronger form for v, once we have introduced the concept of vector space.

The Concept of Functions as Vectors

Before we state the boundary conditions to be satisfied by v and Z at $y = \pm 1$ it is useful to bring in the concept of a linear vector space (Halmos (5)).

Geometrically a linear vector space consists of all the vectors which can be constructed from the basic vectors. The complete set of these basic vectors is called the base and any vector is made up of a sum of components directed along these basic vectors and of a length equal to a multiple of that of the basic vector.

Algebraically a linear space consists of the totality of linear combinations that can be formed out of a set of linearly independent quantities. The complete set of these linearly independent quantities is the base and their number gives the dimension of the space.

If $(\underline{b}_1, \underline{b}_2, \ldots . \underline{b}_N)$ form the base of a vector space, any vector $\underline{V}$ in that space is a linear combination of these basic vectors:

$$\underline{V} = \sum_{j=1}^{N} a_j \underline{b}_j$$

where the a_j' s are real numbers.

The $\underline{b}_j$'s satisfy the orthogonal property

$$\underline{b}_j \cdot \underline{b}_k = l_j \delta_{jk}$$

where l_j is the length of the j^{th} basic vector and δ_{jk} is the Kronecker delta symbol. The dot product is as yet a purely symbolic operation.

Orthogonality insures linear independence since

$$\sum_{j=1}^{N} c_j \underline{b}_j = 0$$

implies $c_j = 0$ for all j's.

We deduce

$$\underline{V} \cdot \underline{b}_j = a_j$$

from the above relations without any need to go further into the significance of the dot product.

An example of a linear space is the three dimensional Euclidean space which is spanned by the three basic vectors $(\underline{b}_1, \underline{b}_2, \underline{b}_3)$ satisfying

$$\underline{b}_j \cdot \underline{b}_k = \delta_{jk}$$

If we denote the $\underline{b}_j$'s by the triads

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1),$$

then any vector $\underline{V} = (a_1, a_2, a_3)$ can be written as

$$\underline{V} = \sum_{j=1}^{3} a_j \underline{b}_j$$

This notation suggests the definition of the dot product by

$$\underline{V} \cdot \underline{U} \equiv \sum_{l=1}^{3} V_l U_l$$

since this would insure

$$\begin{aligned} \underline{b}_1 \cdot \underline{b}_2 &= 1\times0 + 0\times1 + 0\times0 = 0 \\ \underline{b}_1 \cdot \underline{b}_1 &= 1\times1 + 0\times0 + 0\times0 = 1 \qquad \text{etc.} \end{aligned}$$

and

$$\underline{V} \cdot \underline{b}_j = a_1\delta_{1j} + a_2\delta_{2j} + a_3\delta_{3j} = \begin{cases} a_1 & \text{if } j = 1 \\ a_2 & \text{if } j = 2 \\ a_3 & \text{if } j = 3 \end{cases}$$

Geometrically the dot product represents the projection of one vector along the other; algebraically it can be considered as a summation over a variable. Thus

$$\underline{b}_j \cdot \underline{b}_k = \sum_{l=1}^{3} b_{jl} b_{kl}$$

For N dimensions, 3 would be replaced by N but 1 could still be interpreted as a direction in the geometric sense.

A natural generalization of the dot product is

$$\underline{b}_j \cdot \underline{b}_k = \int_\alpha^\beta b_j(x)\, b_k(x)\, dx$$

where x is no longer interpreted as a direction but simply as a continuous variable defined over the interval (α, β). Such a definition of the dot product satisfies all the formal properties that one needs of a dot product, two vectors being orthogonal when

$$\int_\alpha^\beta b_j(x)\, b_k(x)\, dx = 1_j\, \delta jk$$

With such a definition we conceive of a linear vector space as the totality of the functions that can exist over the interval (α, β). Such a space has a base $(s_1, s_2, s_3, \ldots.)$ where N can be taken to be ∞ if this is necessary, so that any function can be written as

$$f(x) = \sum_{j=1}^{\infty} \alpha_j S_j(x)$$

or, in vector notation

$$\underline{f} = \sum_{j=1}^{\infty} \alpha_j \underline{S}_j$$

Functions and vectors having been shown to be formally identical we shall conceive of a function as a vector each time this happens to be useful. We state

1) Any vector (function) $\underline{f}$ in a given space can be expressed in terms of a set of basic vectors s_j's:

$$\underline{f} = \sum_{j=1}^{N} \alpha_j \underline{S}_j$$

2) The component α_j is given by

$$\alpha_j = \int_{-a}^{a} f(x)\ S_j(x)\ dx$$

if the space is defined over the interval

$$-a \leqslant x \leqslant a$$

3) Two vector $\underline{f}$ and $\underline{g}$ are equal only if each of their components are equal:

We know we may write

$$\underline{f} = \Sigma\ \alpha_j \underline{S}_j \qquad \underline{g} = \Sigma\ c_j \underline{S}_j$$

If

$$\underline{f} = \underline{g}$$

this involves

$$\alpha_j = c_j \qquad j = 1,\ 2,\ 3,\ \ldots .$$

We now wish to apply the above notions to our problem. Solving for u we had obtained the solution in term of the functions (sin n'x, cos m'x).

This set of functions forms the base of a space consisting of all the functions, continuous of discontinuous, defined over $-a \leqslant x \leqslant a$ and which vanish at ± a. This space contains in particular the constant 0 but none other.

The basic vectors

$$(\sin n'x,\ \cos m'x)$$

are orthogonal

$$\int_{-a}^{a} \sin n'x \sin N'x\ dx = a\ \delta_{nN}$$

$$\int_{-a}^{a} \cos m'x \cos M'x\ dx = a\,\delta_{mM}$$

$$\int_{-a}^{a} \sin n'x \cos m'x\ dx = 0$$

and are therefore linearly independent.

But this space is too small for our purposes; it contains u since u satisfies

$$u(\pm a) = 0$$

but it cannot contain v and Z which do not necessarily vanish at ± a.

We need a space that contains all functions, continuous or discontinuous, defined over $-a \leqslant x \leqslant a$. This space containing all functions defined over the same interval and which vanish at ± a is a subspace of that space and is completely embodied into it.

The set $(\cos n'x, \sin m'x)$ forms a base of this larger space. These vectors are orthogonal

$$\int_{-a}^{a} \cos n'x \cos N'x \, dx = a_n \, \delta_{nN} \qquad a_n \equiv \begin{cases} \frac{a}{2} & n = o \\ a & n \neq o \end{cases}$$

$$\int_{-a}^{a} \sin m'x \sin M'x \, dx = a \delta_{mM}$$

$$\int_{-a}^{a} \sin m'x \cos n'x \, dx = 0$$

and every function can be expressed in terms of these basic vectors. In particular this space contains all the constants, including 0. It contains as well the set $(\cos m'x, \sin n'x)$ These are given in terms of the basic vectors of the larger space by

$$\cos m'x = \sum_{n=0}^{\infty} P_{mn} \cos n'x$$

$$\sin n'x = \sum_{m=1}^{\infty} Q_{nm} \sin m'x$$

$$P_{mn} \equiv \frac{4}{\pi} (-)^{\frac{m-1}{2}} \cdot \begin{cases} \frac{1}{2m} & n = 0 \\ \frac{(-)^{n/2} m}{m^2 - n^2} & n \neq 0 \end{cases}$$

$$Q_{nm} \equiv \frac{4}{\pi} (-)^{\frac{n+m-1}{2}} \frac{n}{m^2 - n^2}$$

The base we shall use is then

$$(\cos n'x, \quad \sin m'x) \qquad n = 0, 2, 4, \ \ldots$$

$$m = 1, 3, 5, \ \ldots$$

Finally we must remember that we have three intervals over which we wish to define a space, one for each basin. And when we come to satisfy the boundary conditions at $y = \pm\ l_i$, $i = 1, 2, 3$, all the functions will be at the boundary between these various intervals. We must therefore be able to express any function in terms of the basic set that happens to be convenient; but we may only move from a larger space to its subspace. In our case space 1 is the largest, space 2 the smallest. The base of 2 is contained in space 1 or 3.

This restricts the expansion of a function defined over interval 1 or 3 in terms of base 2; we can do this only if we consider its part which is defined over 2. This will be the case for Z where we wish to insure that

$Z_1\ (+\ l_1)$ and $Z_3\ (-\ l_3)$ are equal to $Z_2\ (-\ l_2)$ and $Z_2(+\ l_2)$ respectively

over $-a_2 \leqslant x \leqslant a_2$, the variation of Z_1 and Z_3 over the remaining part of the interval being left undetermined.

Discontinuous functions (having finite discontinuities) are contained in any of the three spaces; $v_1\ (+\ l_1)$ and $v_3\ (-\ l_3)$ are discontinuous but still they are fully contained in their respective spaces.

The Solution for v in Terms of the Basic Vectors: Some Important Remarks

Although we have derived the solution for v and Z from that of u, we will handle only v and Z from now on.

It is neither convenient nor necessary to expand v and Z in terms of their basic vectors in 1, 2, or 3. But in order to gain further insight into our solution we shall write the solution for v without reference to any sea in terms of the basic vectors. This is

$$
\begin{aligned}
V = {} & \left[B_o \cos K_o (y-1) - i A_o \sin K_o (y-1)\right] \sum_{n=0}^{\infty} p_n \cos n' x \\
& + \left[-i B_o \sin K_o (y-1) + A_o \cos K_o (y-1)\right] \sum_{m=1}^{\infty} q_m \sin m' x \\
& + \sum_{m=1}^{\infty} \left(C_m e^{K_m (y-1)} - D_m e^{-K_m (y+1)} \right) \sin m' x + i\,\Omega \sum_{m=1}^{\infty} \sin m' x \sum_{n=2}^{\infty} \frac{Q_{nm}}{n K_n} \left(A_n e^{K_n (y-1)} + B_n e^{-K_n (y+1)} \right) \\
& - \sum_{n=2}^{\infty} \left(A_n e^{K_n (y-1)} - B_n e^{-K_n (y+1)} \right) \cos n' x + i\,\Omega \sum_{n=0}^{\infty} \cos n' x \sum_{m=1}^{\infty} \frac{P_{mn}}{m K_m} \left(C_m e^{K_m (y-1)} + D_m e^{-K_m (y+1)} \right) \qquad (29)
\end{aligned}
$$

where (p_n, q_m) are the components of $(\cosh \lambda' x, \sinh \lambda' x)$; these are given on page 68 ff.

If we neglect the terms affected by the factor $i\,\Omega$ (nde sea) we note two things:

1) The Poincaré waves are already in terms of the basic vectors,

2) The Kelvin waves only will contribute to the constant term.

The amplitude of the vector directed along the basic vector corresponding to $n = 0$ gives the part of the current which is constant across $-a \leqslant x \leqslant a$ for a given value of y. The higher modes corresponding to $j \geqslant 1$ represent the part of the current which varies with x across the section; in the actual expansion it is given by a superposition of even and odd modes which can represent any reasonable function of x.

If we choose a section across which (29) holds, to be the mouth of a sea in which there is motion due to a neighbouring sea, the current due to the external body of water can always be expanded in terms of the basic vectors (cos n'x, sin m'x) and this allows the evaluation of one half of the double infinity of arbitrary constants contained in (29) by equating the two vector expansions representing v at this section (i. e. we satisfy the boundary condition at the mouth). Theorem 1 states that the only constants which wont be affected by a very small factor at the mouth will be (A_o, B_o, A_n, C_m). Identifying the components of the vectors and choosing for the moment the mouth to lie at $y = +1$ (in our actual model it will lie at $y = -1$), we determine B_o explicitely in terms of the constant value of the current while A_n and C_m will be given in terms of the higher modes and B_o or A_o.

When we come to the next boundary to satisfy the extra boundary condition that will allow the evaluation of the latter half of the double infinity of arbitrary constants, A_n and C_m will have been damped out and since they contain the higher harmonics of v this means that these higher harmonics do not affect at all the solution at the next boundary and that (A_o, B_n, D_m) will be given exclusively in terms of the constant part of the current. Once this boundary condition is satisfied (assuming the current or the elevation to be given explicitely there), (29) is given explicitely in terms of the boundary values.

At the mouth the motion will embody all the higher modes of v as well as the fundamental one but at the next boundary the v solution will depend exclusively on the constant value of the current across the mouth while the higher harmonics of this same current are not involved at all. So that at some distance beyond the mouth the solution is completely indifferent to the part of the current which is function of x across the mouth and it depends mainly on the part which is constant across the initial section.

We may therefore restate Theorem 2 and its Corollary in a much stronger form for v:

Theorem: The part of the current which is constant across the mouth will propagate up a nde sea; what is function of x will be damped.

This theorem is intuitive when we realize that a nde sea in the intermediate between a channel of constant depth and a flat rectangular sea. We shall pursue further this analogy by proving later in the thesis that satisfying the boundary conditions for $n = 0$ for the Kelvin waves only and neglecting all the Poincaré waves in a nde sea is equivalent to a zero order of approximation to satisfying the boundary conditions in a channel of constant depth.

One of our basic assumption is that there is perfect reflection of the head; we wish to see if this is always possible.

By perfect reflection we mean that at the mouth and beyond the mouth $y \leqslant -1$, (we return to our previous choice for the position of the mouth) the motion should be given by

$$V = \mathcal{a}_o \sinh[\lambda' x - iK_o(y-1)] \qquad (30)$$

(30) represents the superposition of two Kelvin waves of equal amplitude $\mathcal{a}$ and travelling in opposite directions. We admit of a "zone of confusion" within the sea but not beyond $y \leqslant -1$.

Using the solution for v in a nde sea and applying the boundary conditions

$$v(+1) = 0$$

$$v(-1) = \mathcal{a}_o \sinh[\lambda' x + iK_o 2l]$$

leads to

$$B_o = 0 \qquad n = 0$$

$$B_o p_n - A_n = 0 \qquad n = 2,4,6\ldots$$

$$A_o q_m + C_m = 0 \qquad m = 1,3,5\ldots$$

at $y = +1$ which implies

$$B_o = 0 \qquad A_n = 0 \qquad C_m = -A_o q_m$$

and

$$iA_o \sin 2K_o l = i\mathcal{a}_o \sin 2K_o l \qquad n = 0$$

$$p_n(iA_o \sin 2K_o l) + B_n = p_n(i\mathcal{a}_o \sin 2K_o l) \qquad n = 2,4,6\ldots$$

$$A_o \cos 2K_o l \cdot q_m - D_m = \mathcal{a}_o \cos 2K_o l \qquad m = 1,3,5\ldots$$

at $y = -1$ so that

$$A_o = \mathcal{a}_o \qquad B_n = 0 \qquad D_m = 0$$

We conclude that in a nde sea

1) Perfect reflection always takes place,

2) There is no "zone of confusion": the motion is given everywhere within the sea and beyond by $\mathcal{a}_o \sinh[\lambda'_x - iK_o(y-1)]$ except in the immediate vicinity of the reflecting wall.

This conclusion is again intuitive since a nde sea is very closely related to a channel of constant depth where perfect reflection is always possible. But this seems to clash with Taylor's (15) conclusion that in a semi infinite channel Kelvin waves can be perfectly reflected only if the reflecting wall is located at some specific distance from a given origin.

The sea studied by Taylor does not satisfy the restrictive assumption of narrowness and depth; but by definition it is strictly elongated since the mouth is assumed to lie at $y = -\infty$

Let us for the moment take the mouth at $y = \quad 1'$, $1'$ being some large number.

The condition $v(+1) = 0$, using the full v solution gives

$$B_o p_o + i\Omega \sum_{m=1}^{\infty} \frac{P_{mo}}{mK_m} C_m = 0 \qquad n = 0$$

$$B_o p_n - A_n + i\Omega \sum_{m=1}^{\infty} \frac{P_{mn}}{mK_m} C_m = 0 \qquad n = 2,4,6\ldots$$

$$A_o q_m + C_m + i\Omega \sum_{n=2}^{\infty} \frac{Q_{nm}}{nK_n} A_n = 0 \qquad m = 1,3,5\ldots \qquad (31)$$

We are no longer allowed to set $B_o = 0$ nor $A_o = a_o$ beforehand. These two unknowns should be evaluated by using the boundary condition at $-1'$ because then we would have $2x\,\infty$ equations in $2x\,\infty$ unknowns.

If we insist on setting $B_o = 0$, $A_o = a_o$, the lower boundary condition will yield

$$B_n = D_m = 0$$

i.e. the motion is purely given by a_o inside and outside the sea. This is the situation wished and described by Taylor. At the same time it leaves the system of equations (31) overdeterminate since we have $1x\,\infty$ relations in $1x\,\infty\,-1$ unknowns. Unless the determinant of this system of equations vanishes there is no other solution but the identically zero solution:

$$A_o = C_m = A_n = 0$$

which contradicts the assumption

$$A_o = a_o$$

There is no reason at all for the determinant of (31) to vanish unless we introduce into it some parameter. The obvious choice is 1.

We transform the coordinates so that the origin lies at - 1 ; the head lies at y = + 21

The system (31), assuming $B_0 = 0$, $A_0 = a_0$, becomes

$$p_0 a_0 \sin K_0 l - \Omega \sum_{m=1}^{\infty} \frac{P_{mo}}{mK_m} C_m = 0$$

$$p_n a_0 \sin K_0 l - A_n - \Omega \sum_{m=1}^{\infty} \frac{P_{mn}}{mK_m} C_m = 0$$

$$q_m a_0 \cos K_0 l + C_m + \Omega \sum_{n=2}^{\infty} \frac{Q_{nm}}{nK_n} A_n = 0$$

We redefined the arbitrary constant A_n to be iA_n $(n \neq o)$.

The determinant of the coefficients is

$$\begin{array}{c|ccc|}
 & 0 & \text{odd} & \text{even} \\
0 & p_0 a_0 \sin K_0 l & -\Omega \dfrac{P_{mo}}{mK_m} & 0 \\
\text{odd} & q_m a_0 \cos K_0 l & \delta_{mM} & \Omega \dfrac{Q_{nm}}{nK_n} \\
\text{even} & p_n a_0 \sin K_0 l & -\Omega \dfrac{P_{mn}}{mK_m} & -\delta_{nN}
\end{array}$$

Where 0 denotes column or row 0 , m, odd columns and rows and n, even columns and rows, excluding 0.

If we insert the explicit expression for P_{mn}, Q_{mn}, p_n and q_m, and if we perform the following operations:

1) Remove the common factor a_0 from column 0

2) Take as Taylor, $\lambda' a = \frac{\lambda\pi}{2}$

3) Multiply row 0 by the constant 2,

4) Remove the common factor $\frac{4}{\pi} \lambda \cos K_0 l$ from column 0,

5) Multiply all odd rows and columns by $(-)^{\frac{m-1}{2}}$, all even rows and columns

by $(-)^{\frac{n}{2}}$

6) Remove the common factor $\frac{\Omega}{K_j}\frac{4}{\pi}$ from all even and odd columns,

7) Multiply all odd columns by $\sinh\frac{\lambda\pi}{2}$, all even columns by $\cosh\frac{\lambda\pi}{2}$

8) Divide all odd rows by $\cosh\frac{\lambda\pi}{2}$, all even rows by $\sinh\frac{\lambda\pi}{2}$,

We obtain

$$\begin{array}{c|ccc|}
 & 0 & \text{odd} & \text{even} \\
0 & \frac{\operatorname{tg} K_o l}{\lambda^2} & -\frac{1}{m^2} & 0 \\
\text{odd} & \frac{1}{\lambda^2+m^2} & \frac{4}{\pi}\cdot\frac{K_m}{\Omega}\cdot \operatorname{tgh}\frac{\lambda\pi}{\lambda}\cdot\delta_{mM} & \frac{1}{m^2-n^2} \\
\text{even} & \frac{\operatorname{tg} K_o l}{\lambda^2+n^2} & \frac{1}{n^2-m^2} & -\frac{4}{\pi}\cdot\frac{K_n}{\Omega}\cdot \operatorname{cotgh}\frac{\lambda\pi}{2}\cdot\delta_{nN}
\end{array}$$

which is the determinant derived by Taylor but expressed in our own notation making due allowance to the fact that our a_o corresponds to $a_{o/i}$ in Taylor's notation and that our frame of reference is left handed while his is right handed. Such a determinant gives the values of l for which the system (31) will have a solution that can be expressed in terms of a_o only.

As long as l ' is finite, l is a paradoxical quantity since it is found in a way that makes it completely independent of l ' and it depends exclusively on our choice of an origin.

l can be meaningful only if $l' \to \infty$; then we wish to insure that a_o prevails over the whole of the semi infinite channel.

If l ' is finite, l is meaningless and we need to solve the full 2x ∞ linear **equations to extract values for A_o and B_o. In this case A_o is different from a_o** but the Poincaré waves B_n and D_m are such that they balance A_o and B_o to produce a_o at the mouth.

Perfect reflection takes place as long as K_j is real for all j's. But over the sea, between the mouth and the head, there is a "zone of confusion" where B_o exists along A_o.

As the sea becomes more and more nde, $\Omega \to 0$ $B_o \to 0$ and $A_o \to a_o$.

The above considerations allow us to state the general theorem:

Theorem: If K_j is real for all j's, a Kelvin wave can always be perfectly reflected in a rotating rectangular sea of constant depth of any length.

The Boundary Conditions

At $y_1 = -l_1$

$$v_1(-l_1) = f(x) = \sum_{j=0}^{\infty} \theta_j S_j(x) \qquad -a_1 \leqslant x \leqslant a_1 \qquad (32)$$

At $y_1 = +l_1$, $y_2 = -l_2$

$$v_1(+l_1) \cdot \frac{h_1}{h_2} = \begin{cases} 0 & a_2 < |x| \leqslant a_1 \\ \\ v_2(-l_2) & -a_2 \leqslant x \leqslant a_2 \end{cases} \qquad (33)$$

$$Z_1(+l_1) = Z_2(-l_2) \qquad -a_2 \leqslant x \leqslant a_2 \qquad (34)$$

At $y_2 = +l_2$, $y_3 = -l_3$

$$v_3(-l_3) \cdot \frac{h_3}{h_2} = \begin{cases} 0 & a_2 < |x| \leqslant a_3 \\ \\ v_2(+l_2) & -a_2 \leqslant x \leqslant a_2 \end{cases} \qquad (35)$$

$$Z_3(-l_3) = Z_2(+l_2) \qquad -a_2 \leqslant x \leqslant a_2 \qquad (36)$$

At $y_3 = +l_3$ $\quad v_3(+l_3) = 0 \qquad -a_3 \leqslant x \leqslant a_3 \qquad (37)$

The boundary condition at the mouth has already been discussed.

The v boundary condition at the junctions of the basins indicates that $v_1\,(+\,l_1)$ and $v_3\,(-\,l_3)$ are equal to a vector with the properties described on the right hand side. This will involve evaluating such a vector in space 1 and 3. The Z boundary condition restricts the value of Z_1 and Z_3 only over the interval $-a_2 \leqslant x \leqslant a_2$; we then have to express $Z_1\,(+\,l_1)$ and $Z_3\,(-\,l_3)$ in terms of the basic vectors in space 2 and equate them to $Z_2\,(\mp\,l_2)$.

We therefore need

1) $v_1\,(+\,l_1)$ and $v_3\,(+\,l_3)$ in their respective spaces.

2) A vector which we shall denote by $V\,(+\,l_2)$ with the properties described by (33) and (35) in space 1 and 3.

3) $Z_1\,(+\,l_1)$, $Z_3\,(-\,l_3)$ and $Z_2\,(\pm\,l_2)$ in space 2.

These quantities are, using solutions (26) and (27), for k = 1 or 3:

$$
\begin{aligned}
v_k(\pm l_k) = {} & \begin{bmatrix} B_o^{\textcircled{k}} \\ B_o^{\textcircled{k}} \cos 2K_o^{\textcircled{k}} l_k + i A_o^{\textcircled{k}} \sin 2K_o^{\textcircled{k}} l_k \end{bmatrix} \cdot \sum_{n_k=0}^{\infty} p_{n_k}^{\textcircled{k}} \cos n'_k x \\
& + \begin{bmatrix} A_o^{\textcircled{k}} \\ A_o^{\textcircled{k}} \cos 2K_o^{\textcircled{k}} l_k + i B_o^{\textcircled{k}} \sin 2K_o^{\textcircled{k}} l_k \end{bmatrix} \cdot \sum_{m_k=1}^{\infty} q_{m_k}^{\textcircled{k}} \sin m'_k x \\
& + \sum_{n_k=1}^{\infty} \cos n'_k x \begin{pmatrix} -A_{n_k}^{\textcircled{k}} \\ B_{n_k}^{\textcircled{k}} \end{pmatrix} + \sum_{m_k=1}^{\infty} \sin m'_k x \begin{pmatrix} C_{m_k}^{\textcircled{k}} \\ -D_{m_k}^{\textcircled{k}} \end{pmatrix}
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
V(\pm l_2) = {} & \begin{bmatrix} B_o^{\textcircled{2}} \\ B_o^{\textcircled{2}} \cos 2K_o^{\textcircled{2}} l_2 + i A_o^{\textcircled{2}} \sin 2K_o^{\textcircled{2}} l_2 \end{bmatrix} \cdot \sum_{n_k=0}^{\infty} r_{n_k}^{\textcircled{2}} \cos n'_k x \\
& + \begin{bmatrix} A_o^{\textcircled{2}} \\ A_o^{\textcircled{2}} \cos 2K_o^{\textcircled{2}} l_2 + i B_o^{\textcircled{2}} \sin 2K_o^{\textcircled{2}} l_2 \end{bmatrix} \cdot \sum_{m_k=1}^{\infty} t_{m_k}^{\textcircled{2}} \sin m'_k x \\
& + \frac{a_2}{a_k} \left\{ \sum_{n_k=2}^{\infty} \cos n'_k x \sum_{n_2=2}^{\infty} P_{n_k n_2} \begin{pmatrix} -A_{n_2}^{\textcircled{2}} \\ B_{n_2}^{\textcircled{2}} \end{pmatrix} + \sum_{m_k=1}^{\infty} \sin m'_k x \sum_{m_2=1}^{\infty} Q_{m_k m_2} \begin{pmatrix} C_{m_2}^{\textcircled{2}} \\ -D_{m_2}^{\textcircled{2}} \end{pmatrix} \right\}
\end{aligned}
\tag{39}
$$

$$\begin{aligned}
Z_k(\pm l_k) = \left(\frac{h_k}{g}\right)^{\frac{1}{2}} \left\{ \begin{bmatrix} A_o^{\textcircled{k}} \\ A_o^{\textcircled{k}} \cos 2K_o^{\textcircled{k}} l_k + i\, B_o^{\textcircled{k}} \sin 2K_o^{\textcircled{k}} l_k \end{bmatrix} \sum_{n_2=0}^{\infty} u_{n_2}^{\textcircled{k}} \cos n_2' x \right. \\
\left. + \begin{bmatrix} B_o^{\textcircled{k}} \\ B_o^{\textcircled{k}} \cos 2K_o^{\textcircled{k}} l_k + i A_o^{\textcircled{k}} \sin 2K_o^{\textcircled{k}} l_k \end{bmatrix} \sum_{m_2=1}^{\infty} v_{m_2}^{\textcircled{k}} \sin m_2' x \right\} \\
- \propto_k \left\{ \sum_{n_2=0}^{\infty} \cos n_2' x \left[-i\sigma \sum_{n_k=2}^{\infty} \frac{P_{n_k n_2}}{K_{n_k}} \begin{pmatrix} A_{n_k}^{\textcircled{k}} \\ B_{n_k}^{\textcircled{k}} \end{pmatrix} + 2\omega_k \sum_{m_k=1}^{\infty} \frac{P_{m_k n_2}}{m_k} \begin{pmatrix} C_{m_k}^{\textcircled{k}} \\ -D_{m_k}^{\textcircled{k}} \end{pmatrix} \right] \right. \\
\left. + \sum_{m_2=1}^{\infty} \sin m_2' x \left[i\sigma \sum_{m_k=1}^{\infty} \frac{Q_{m_k m_2}}{K_{m_k}} \begin{pmatrix} C_{m_k}^{\textcircled{k}} \\ D_{m_k}^{\textcircled{k}} \end{pmatrix} + 2\omega_k \sum_{n_k=2}^{\infty} \frac{Q_{n_k m_2}}{n_k} \begin{pmatrix} A_{n_k}^{\textcircled{k}} \\ -B_{n_k}^{\textcircled{k}} \end{pmatrix} \right] \right\}
\end{aligned} \tag{40}$$

$$\begin{aligned}
Z_2(\pm l_2) = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} \left\{ \begin{bmatrix} A_o^{\textcircled{2}} \\ A_o^{\textcircled{2}} \cos 2K_o^{\textcircled{2}} l_2 + i\, B_o^{\textcircled{2}} \sin 2K_o^{\textcircled{2}} l_2 \end{bmatrix} \sum_{n_2=0}^{\infty} p_{n_2}^{\textcircled{2}} \cos n_2' x \right. \\
\left. + \begin{bmatrix} B_o^{\textcircled{2}} \\ B_o^{\textcircled{2}} \cos 2K_o^{\textcircled{2}} l_2 + i A_o^{\textcircled{2}} \sin 2K_o^{\textcircled{2}} l_2 \end{bmatrix} \sum_{m_2=1}^{\infty} q_{m_2}^{\textcircled{2}} \sin m_2' x \right. \\
- \propto_2 \left\{ -i\sigma \sum_{n_2=2}^{\infty} \frac{\cos n_2' x}{K_{n_2}} \begin{pmatrix} A_{n_2}^{\textcircled{2}} \\ B_{n_2}^{\textcircled{2}} \end{pmatrix} + 2\omega_2 \sum_{n_2=0}^{\infty} \cos n_2' x \left[\sum_{m_2=1}^{\infty} \frac{P_{m_k n_2}}{m_2} \begin{pmatrix} C_{m_2}^{\textcircled{2}} \\ -D_{m_2}^{\textcircled{2}} \end{pmatrix} \right] \right. \\
\left. + \sum_{m_2=1}^{\infty} \sin m_2' x \left[\frac{i\sigma}{K_{m_2}} \begin{pmatrix} C_{m_2}^{\textcircled{2}} \\ D_{m_2}^{\textcircled{2}} \end{pmatrix} + 2\omega_2 \sum_{n_2=2}^{\infty} \frac{Q_{n_2 m_2}}{n_2} \begin{pmatrix} A_{n_2}^{\textcircled{2}} \\ -B_{n_2}^{\textcircled{2}} \end{pmatrix} \right] \right\}
\end{aligned} \tag{41}$$

Where

$$p_{o}^{\textcircled{k}} \equiv \frac{1}{2a_k}\int_{-a_k}^{a_k} \cosh \lambda'_k x \, dx = \frac{\sinh (\lambda'_k a_k)}{(\lambda'_k a_k)}$$

$$p_{n_k}^{\textcircled{k}} \equiv \frac{1}{a_k}\int_{-a_k}^{a_k} \cosh \lambda'_k x \, \cos n'_k x \, dx = \frac{8}{\pi^2} \cdot (\lambda'_k a_k) \cdot (-)^{\frac{n_k}{2}} \cdot \frac{\sinh (\lambda'_k a_k)}{(\lambda_k^2 + n_k^2)}$$

$$q_{m_k}^{\textcircled{k}} \equiv \frac{1}{a_k}\int_{-a_k}^{a_k} \sinh \lambda'_k x \, \sin m'_k x \, dx = \frac{8}{\pi^2} \cdot (\lambda'_k a_k) \cdot (-)^{\frac{m_k - 1}{2}} \cdot \frac{\cosh (\lambda'_k a_k)}{(\lambda_k^2 + m_k^2)}$$

$$r_{o_k}^{\textcircled{2}} \equiv \frac{1}{2a_k}\int_{-a_2}^{a_2} \cosh \lambda'_2 x \, dx = \frac{\sinh (\lambda'_2 a_2)}{(\lambda'_2 a_k)}$$

$$r_{n_k}^{\textcircled{2}} \equiv \frac{1}{a_k}\int_{-a_2}^{a_2} \cosh \lambda'_2 x \cos n'_k x \, dx = \frac{4}{\pi} \frac{\left\{\frac{\lambda_2 a_k}{a_2} \cdot \sinh \lambda'_2 a_2 \cdot \cos n'_k a_2 + n_k \cdot \cosh \lambda'_2 a_2 \cdot \sin m'_k a_2\right\}}{\left\{\left(\frac{\lambda_2 a_k}{a_2}\right)^2 + n_k^2\right\}}$$

$$t_{m_k}^{\textcircled{2}} \equiv \frac{1}{a_k}\int_{-a_2}^{a_2} \sinh \lambda'_2 x \, \sin m'_k \, x \, dx = \frac{4}{\pi} \frac{\left\{\frac{\lambda_2 a_k}{a_2} \cdot \cosh \lambda'_2 a_2 \cdot \sin m'_k a_2 - m_k \sinh \lambda'_2 a_2 \cdot \cos m'_k a_2\right\}}{\left\{\left(\frac{\lambda_2 a_k}{a_2}\right)^2 + m_k^2\right\}}$$

$$P_{j_k o_i} \equiv \frac{1}{2a_i}\int_{-a_i}^{a_i} \cos j_k' x \, dx = \frac{2}{\pi} \cdot \frac{a_k}{a_i} \cdot \frac{\sin(j_k' a_i)}{j_k} = \frac{2}{\pi} \; \frac{(-)^{\frac{m-1}{2}}}{m} \quad \text{for} \begin{cases} i = k \\ j_k = m \end{cases}$$

$$P_{j_k n_i} \equiv \frac{1}{a_i}\int_{-a_i}^{a_i} \cos j_k' x \cos n_i' x \, dx = \frac{4}{\pi} \cdot \frac{a_k}{a_i} \cdot (-)^{\frac{n_i}{2}} \; \frac{j_k \sin(j_k' a_i)}{\left[j_k^2 - \left(\frac{n_i a_k}{a_i}\right)^2 \right]}$$

$$= \frac{4}{\pi} (-)^{\frac{n+m-1}{2}} \frac{m}{m^2 - n^2} \quad \text{for} \begin{cases} i = k \\ j_k = m \\ n_i = n \end{cases}$$

$$Q_{j_k m_i} \equiv \frac{1}{a_i}\int_{-a_i}^{a_i} \sin j_k' x \sin m_i' x \, dx = -\frac{4}{\pi} \cdot \frac{a_k}{a_i} \; (-)^{\frac{m_i - 1}{2}} \cdot \frac{j_k \cos(j_k' a_i)}{\left[j_k^2 - \left(\frac{m_i a_k}{a_i}\right)^2 \right]}$$

$$= -\frac{4}{\pi} (-)^{\frac{n+m-1}{2}} \frac{n}{n^2 - m^2} \quad \text{for} \begin{cases} i = k \\ j_k = n \\ m_i = m \end{cases}$$

$$u_{o_2}^{\textcircled{k}} \equiv \frac{1}{2a_2}\int_{-a_2}^{a_2} \cosh \lambda_k' x \, dx = \frac{\sinh(\lambda_k' a_2)}{(\lambda_k' a_2)}$$

$$u_{n_2}^{\textcircled{k}} \equiv \frac{1}{a_2}\int_{-a_2}^{a_2} \cosh \lambda_k' x \cos n_2' x \, dx = \frac{8}{\pi^2} \cdot (\lambda_k' a_2) \cdot (-)^{\frac{n_2}{2}} \cdot \frac{\sinh(\lambda_k' a_2)}{\left[n_2^2 + \left(\frac{\lambda_k a_2}{a_k}\right)^2\right]}$$

$$v_{m_2}^{\textcircled{k}} \equiv \frac{1}{a_2}\int_{-a_2}^{a_2} \sinh \lambda_k' x \sin m_2' x \, dx = \frac{8}{\pi^2} \cdot (\lambda_k' a_2) \cdot (-)^{\frac{m-1}{2}} \cdot \frac{\cosh(\lambda_k' a_2)}{\left[m_2^2 + \left(\frac{\lambda_k a_2}{a_k}\right)^2\right]}$$

We write out the linear equations coming from the v boundary conditions then those arising from the conditions on Z.

$$\left\{\begin{array}{ll} (B_o^{\textcircled{1}}\cos 2K_o^{\textcircled{1}} l_1 + i\,A_o^{\textcircled{1}}\sin 2K_o^{\textcircled{1}} l_1)\,p_{o_1}^{\textcircled{1}} = \theta_o & \\ (B_o^{\textcircled{1}}\cos 2K_o^{\textcircled{1}} l_1 + i\,A_o^{\textcircled{1}}\sin 2K_o^{\textcircled{1}} l_1)\,p_{n_1}^{\textcircled{1}} + B_{n_1}^{\textcircled{1}} = \theta_{n_1} & n_1 = 2,4,6\ldots \\ (A_o^{\textcircled{1}}\cos 2K_o^{\textcircled{1}} l_1 + i\,B_o^{\textcircled{1}}\sin 2K_o^{\textcircled{1}} l_1)\,q_{m_1}^{\textcircled{1}} - D_{m_1}^{\textcircled{1}} = \theta_{m_1} & m_1 = 1,3,5\ldots \end{array}\right\} \qquad (42)$$

$$\left\{\begin{array}{l} \frac{h_1}{h_2}\cdot B_o^{\textcircled{1}}p_{o_1}^{\textcircled{1}} = (B_o^{\textcircled{2}}\cos 2K_o^{\textcircled{1}} l_2 + i\,A_o^{\textcircled{2}}\sin 2K_o^{\textcircled{2}} l_2)\,r_{o_1}^{\textcircled{2}} \\ \frac{h_1}{h_2}\{B_o^{\textcircled{1}}p_{n_1}^{\textcircled{1}} - A_{n_1}^{\textcircled{1}}\} = (B_o^{\textcircled{2}}\cos 2K_o^{\textcircled{2}} l_2 + i\,A_o^{\textcircled{2}}\sin 2K_o^{\textcircled{2}} l_2)\,r_{n_1}^{\textcircled{2}} + \frac{a_2}{a_1}\cdot \sum_{n_2=2}^{\infty} P_{n_1 n_2} B_{n_2}^{\textcircled{2}} \\ \frac{h_1}{h_2}\{A_o^{\textcircled{1}}q_{m_1}^{\textcircled{1}} + C_{m_1}^{\textcircled{1}}\} = (A_o^{\textcircled{2}}\cos 2K_o^{\textcircled{2}} l_2 + i\,B_o^{\textcircled{2}}\sin 2K_o^{\textcircled{2}} l_2)\,t_{m_1}^{\textcircled{2}} - \frac{a_2}{a_1}\cdot \sum_{m_2=1}^{\infty} Q_{m_1 m_2} D_{m_2}^{\textcircled{2}} \end{array}\right\} \qquad (43)$$

$$\left\{\begin{array}{l} \frac{h_3}{h_2}\{(B_o^{\textcircled{3}}\cos 2K_o^{\textcircled{3}} l_3 + i\,A_o^{\textcircled{3}}\sin 2K_o^{\textcircled{3}} l_3)\,p_{o_3}^{\textcircled{3}}\} = B_o^{\textcircled{2}} r_{o_3}^{\textcircled{2}} \\ \frac{h_3}{h_2}\{(B_o^{\textcircled{3}}\cos 2K_o^{\textcircled{3}} l_3 + i\,A_o^{\textcircled{3}}\sin 2K_o^{\textcircled{3}} l_3)\,p_{n_3}^{\textcircled{3}} + B_{n_3}^{\textcircled{3}}\} = B_o^{\textcircled{2}} r_{n_3}^{\textcircled{2}} - \frac{a_2}{a_3} \sum_{n_3=2}^{\infty} P_{n_3 n_2} A_{n_2}^{\textcircled{2}} \\ \frac{h_3}{h_2}\{(A_o^{\textcircled{3}}\cos 2K_o^{\textcircled{3}} l_3 + i\,B_o^{\textcircled{3}}\sin 2K_o^{\textcircled{3}} l_3)\,q_{m_3}^{\textcircled{3}} - D_{m_3}^{\textcircled{3}}\} = A_o^{\textcircled{2}} t_{m_3}^{\textcircled{2}} + \frac{a_2}{a_3} \sum_{m_2=1}^{\infty} Q_{m_3 m_2}\, C_{m_2}^{\textcircled{2}} \end{array}\right\} \qquad (44)$$

$$\left\{\begin{array}{l} B_o^{\textcircled{3}} = 0 \\ A_{n_3}^{\textcircled{3}} = 0 \\ C_{m_3}^{\textcircled{3}} = -\,A_o^{\textcircled{3}} q_{m_3}^{\textcircled{3}} \end{array}\right\} \qquad (45)$$

$$\left\{\begin{aligned}
&\left(\frac{h_1}{g}\right)^{\frac{1}{2}} A_o^{\textcircled{1}} u_{o_2}^{\textcircled{1}} - 2\omega_1\alpha_1 \sum_{m_1=1}^{\infty} \frac{P_{m_1 o}}{m_1} C_{m_1}^{\textcircled{1}} + i\sigma\alpha_1 \sum_{n_1=2}^{\infty} \frac{P_{n_1 o}}{K_{n_1}} A_{n_1}^{\textcircled{1}} \\
&\quad = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} \left[\left(A_o^{\textcircled{2}} \cos 2K_o^{\textcircled{2}} l_2 + iB_o^{\textcircled{2}} \sin 2K_o^{\textcircled{2}} l_2\right) p_o^{\textcircled{2}}\right] + 2\omega_2\alpha_2 \sum_{m_2=1}^{\infty} \frac{P_{m_2 o}}{m_2} D_{m_2}^{\textcircled{2}} \\
&\left(\frac{h_1}{g}\right)^{\frac{1}{2}} A_o^{\textcircled{1}} u_{n_1}^{\textcircled{1}} + i\sigma\alpha_1 \sum_{n_1=2}^{\infty} \frac{P_{n_1 n_2}}{K_{n_1}} A_{n_1}^{\textcircled{1}} - 2\omega_1\alpha_1 \sum_{m_1=1}^{\infty} \frac{P_{m_1 n_2}}{m_1} C_{m_1}^{\textcircled{1}} \\
&\quad = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} \left[\left(A_o^{\textcircled{2}} \cos 2K_o^{\textcircled{2}} l_2 + iB_o^{\textcircled{2}} \sin 2K_o^{\textcircled{2}} l_2\right) p_{n_2}^{\textcircled{2}}\right] + i\sigma\alpha_2 \frac{B_{n_2}}{K_{n_2}} + 2\omega_2\alpha_2 \sum_{m_2=1}^{\infty} \frac{P_{m_2 n_2}}{m_2} D_{m_2}^{\textcircled{2}} \\
&\left(\frac{h_1}{g}\right)^{\frac{1}{2}} B_o^{\textcircled{1}} v_{m_2}^{\textcircled{1}} - i\sigma\alpha_1 \sum_{m_1=1}^{\infty} \frac{Q_{m_1 m_2}}{K_{m_1}} C_{m_1}^{\textcircled{1}} - 2\omega_1\alpha_1 \sum_{n_1=2}^{\infty} \frac{Q_{n_1 m_2}}{n_1} A_{n_1}^{\textcircled{1}} \\
&\quad = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} \left[\left(B_o^{\textcircled{2}} \cos 2K_o^{\textcircled{2}} l_2 + iA_o^{\textcircled{2}} \sin 2K_o^{\textcircled{2}} l_2\right) q_{m_2}^{\textcircled{2}}\right] - \frac{i\sigma\alpha_2}{K_{m_2}} D_{m_2}^{\textcircled{2}} + 2\omega_2\alpha_2 \sum_{n_2=2}^{\infty} \frac{Q_{n_2 m_2}}{n_2} B_{n_2}^{\textcircled{2}}
\end{aligned}\right\} \tag{46}$$

$$\left\{\begin{aligned}
&\left(\frac{h_3}{g}\right)^{\frac{1}{2}} \left(A_o^{\textcircled{3}} \cos 2K_o^{\textcircled{3}} l_3 + iB_o^{\textcircled{3}} \sin 2K_o^{\textcircled{3}} l_3\right) u_{o_2}^{\textcircled{3}} + i\sigma\alpha_3 \sum_{n_3=2}^{\infty} \frac{P_{n_3 o}}{K_{n_3}} B_{n_3}^{\textcircled{3}} + 2\omega_3\alpha_3 \sum_{m_3=1}^{\infty} \frac{P_{m_3 o}}{m_3} D_{m_3}^{\textcircled{3}} \\
&\quad = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} A_o^{\textcircled{2}} p_{o_2}^{\textcircled{2}} - 2\omega_2\alpha_2 \sum_{m_2=1}^{\infty} \frac{P_{m_2 o}}{m_2} C_{m_2}^{\textcircled{2}} \\
&\left(\frac{h_3}{g}\right)^{\frac{1}{2}} \left(A_o^{\textcircled{3}} \cos 2K_o^{\textcircled{3}} l_3 + iB_o^{\textcircled{3}} \sin 2K_o^{\textcircled{3}} l_3\right) u_{n_2}^{\textcircled{3}} + i\sigma\alpha_3 \sum_{n_3=2}^{\infty} \frac{P_{n_3 n_2}}{K_{n_3}} B_{n_3}^{\textcircled{3}} + 2\omega_3\alpha_3 \sum_{m_3=1}^{\infty} \frac{P_{m_3 n_2}}{m_3} D_{m_3}^{\textcircled{3}} \\
&\quad = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} A_o^{\textcircled{2}} p_{n_2}^{\textcircled{2}} + i\sigma\alpha_2 \frac{A_{n_2}^{\textcircled{2}}}{K_{n_2}} - 2\omega_2\alpha_2 \sum_{m_2=1}^{\infty} \frac{P_{m_2 n_2}}{m_2} C_{m_2}^{\textcircled{2}} \\
&\left(\frac{h_3}{g}\right)^{\frac{1}{2}} \left(B_o^{\textcircled{3}} \cos 2K_o^{\textcircled{3}} l_3 + iA_o^{\textcircled{3}} \sin 2K_o^{\textcircled{3}} l_3\right) v_{m_2}^{\textcircled{3}} - i\sigma\alpha_3 \sum_{m_3=1}^{\infty} \frac{Q_{m_3 m_2}}{K_{m_3}} D_{m_3}^{\textcircled{3}} + 2\omega_3\alpha_3 \sum_{n_3=2}^{\infty} \frac{Q_{n_3 m_2}}{n_3} B_{n_3}^{\textcircled{3}} \\
&\quad = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} B_o^{\textcircled{2}} q_{m_2}^{\textcircled{2}} - i\sigma\alpha_2 \frac{C_{m_2}^{\textcircled{2}}}{K_{m_2}} - 2\omega_2\alpha_2 \sum_{n_2=2}^{\infty} \frac{Q_{n_2 m_2}}{n_2} A_{n_2}^{\textcircled{2}}
\end{aligned}\right\} \tag{47}$$

The Solution of the Linear Equations

Equations (42) to (47) represent 6x ∞ linear relations between 6x ∞ unknowns, all of these complex.

Equations (42) and (45) show that $A_m^{③}$, $C_m^{③}$, $B_n^{①}$, $D_m^{①}$ are really not involved in the system and may be evaluated once the Kelvin waves are known. A twofold infinity of unknowns is already eliminated.

By the way this makes the theorem on v more obvious since it can be seen that θ_n, and θ_m, the higher harmonics of the currents, are not involved in the equations giving the arbitrary constants in 2. Only $B_n^{①}$, and $D_m^{①}$, which describe the current in the vicinity of the mouth depend on these quantities.

We use (43) and (44) to express the arbitrary constants in 1 and 3 in terms of sea 2.

We end up with a system of 2x ∞ linear relations between 2x ∞ unknowns:

$$\begin{pmatrix} B_o^{\textcircled{2}} \\ B_o^{\textcircled{2}} \cos 2K_o^{\textcircled{2}} l_2 + i A_o^{\textcircled{2}} \sin 2K_o^{\textcircled{2}} l_2 \end{pmatrix} (-)^{\frac{k-1}{2}} i \left[\frac{\sigma \alpha_k h_2}{h_k} \sum_{n_k=2}^{\infty} \frac{P_{n_k n}}{K_{n_k}} \cdot r_{n_k}^{\textcircled{2}} - G^{\textcircled{k}} \left\{ \operatorname{cotg}(2K_o^{\textcircled{k}} l_k) \left[\left(\frac{h_k}{g}\right)^{\frac{1}{2}} u_n^{\textcircled{k}} \right. \right. \right.$$

$$\left. \left. \left. + \sigma \alpha_k \sum_{n_k=2}^{\infty} \frac{P_{n_k n}}{K_{n_k}} \, p_{n_k}^{\textcircled{k}} \right\} \right] + 2\omega_k \alpha_k \sum_{m_k=1}^{\infty} \frac{P_{n_k n}}{m_k} \, q_{n_k}^{\textcircled{k}} \right]$$

$$+ \begin{pmatrix} A_o^{\textcircled{2}} \\ A_o^{\textcircled{2}} \cos 2K_o^{\textcircled{2}} l_2 + i B_o^{\textcircled{2}} \sin 2K_o^{\textcircled{2}} l_2 \end{pmatrix} \left[\left(\frac{h_2}{g}\right)^{\frac{1}{2}} p_n^{\textcircled{2}} + \frac{2\omega_k \alpha_k \, h_2}{h_k} \sum_{m_k=1}^{\infty} \frac{P_{m_k n}}{m_k} \, t_{m_k}^{\textcircled{2}} \right]$$

$$+ \frac{\sigma \alpha_2}{K_n} \begin{pmatrix} A_n \\ B_n \end{pmatrix} + \sigma \alpha_k \frac{h_2 a_2}{h_k a_k} \sum_{N=2}^{\infty} \begin{pmatrix} A_n \\ B_n \end{pmatrix} \sum_{n_k=2}^{\infty} \frac{P_{n_k n} P_{n_k N}}{K_{n_k}}$$

$$- \sum_{m=1}^{\infty} \begin{pmatrix} C_m \\ -D_m \end{pmatrix} \left[2\omega_2 \alpha_2 \frac{P_{mn}}{m} - 2\omega_k \alpha_k \frac{h_2 a_2}{h_k a_k} \sum_{m_k=1}^{\infty} \frac{P_{m_k n} \, Q_{m_k m}}{m_k} \right].$$

$$= \begin{pmatrix} 0 \\ \dfrac{F^{\textcircled{1}}}{i \sin 2K_o^{\textcircled{1}} l_1} \left[\left(\dfrac{h_1}{g}\right)^{\frac{1}{2}} u_n^{\textcircled{1}} + 2\omega_1 \alpha_1 \sum_{m_1=1}^{\infty} \dfrac{P_{m_1 n_m}}{m_1} \, q_{m_1}^{\textcircled{1}} \right. \end{pmatrix} . \qquad n = 0,2,4,6 \ldots.$$

$$\begin{pmatrix} B_o^{\textcircled{2}} \\ B_o^{\textcircled{2}}\cos 2K_o^{\textcircled{2}}l_2 + iA_o^{\textcircled{2}}\sin 2K_o^{\textcircled{2}}l_2 \end{pmatrix}\left[\left(\frac{h_2}{g}\right)^{\frac{1}{2}} q_m^{\textcircled{2}} + G^{\textcircled{k}}\left\{-\left(\frac{h_k}{g}\right)^{\frac{1}{2}} v_m^{\textcircled{k}} + \sigma\alpha_k \operatorname{cotg}(2K_o^{\textcircled{k}}l_k) \cdot \sum_{m_k=1}^{\infty} \frac{Q_{m_k m}}{K_{m_k}} q_{m_k}^{\textcircled{k}}\right.\right.$$

$$\left.\left. + 2\omega_k\alpha_k \sum_{n_k=2}^{\infty} \frac{Q_{n_k m}}{n_k} \cdot p_{n_k}^{\textcircled{k}}\right\} - 2\omega_k\alpha_k \frac{h_2}{h_k} \sum_{n_k=2}^{\infty} \frac{Q_{n_k m}}{n_k} r_{n_k}^{\textcircled{2}}\right]$$

$$+ (-)^{\frac{k-1}{2}} \begin{pmatrix} A_o^{\textcircled{2}} \\ A_o^{\textcircled{2}}\cos 2K_o^{\textcircled{2}}l_2 + iB_o^{\textcircled{2}}\sin 2K_o^{\textcircled{2}}l_2 \end{pmatrix} i\sigma\alpha_k \frac{h_2}{h_k} \sum_{m_k=1}^{\infty} \frac{Q_{m_k m}}{K_{m_k}} t_{n_k}^{\textcircled{2}}$$

$$- i\sigma\left[\frac{\alpha_2}{K_m}\begin{pmatrix} C_m \\ D_m \end{pmatrix} + \alpha_k \frac{h_2 a_2}{h_k a_k} \sum_{M=1}^{\infty} \begin{pmatrix} C_M \\ D_M \end{pmatrix} \sum_{m_k=1}^{\infty} \frac{Q_{m_k M}\, Q_{m_k m}}{K_{m_k}}\right]$$

$$+ i \sum_{n=2}^{\infty} \begin{pmatrix} A_n \\ B_n \end{pmatrix}\left[2\omega_2\alpha_2 \frac{Q_{nm}}{n} - 2\omega_k\alpha_k \frac{h_2 a_2}{h_k a_k} \sum_{n_k=2}^{\infty} \frac{P_{n_k n}\, Q_{n_k m}}{n_k}\right]$$

$$= \begin{pmatrix} 0 \\ \dfrac{\sigma\alpha_1 F^{\textcircled{1}}}{\sin 2K_o^{\textcircled{1}}l_1} \displaystyle\sum_{m_1=1}^{\infty} \frac{Q_{m_1 m}}{K_{m_1}} q_{m_1}^{\textcircled{1}} \end{pmatrix} \qquad m = 1,3,5\ldots.$$

$$F^{\textcircled{1}} \equiv \frac{\theta_o}{p_o^{\textcircled{1}}} \qquad G^{\textcircled{1}} \equiv \frac{r_{o_1}^{\textcircled{2}}}{p_o^{\textcircled{1}}} \cdot \frac{h_2}{h_1} \qquad G^{\textcircled{3}} \equiv \frac{r_{o_3}^{\textcircled{2}}}{p_o^{\textcircled{3}}} \cdot \frac{h_2}{h_3}$$

k stands for 1 or 3. The index 2 has been dropped and the equation for $n = 0$ is contained in the above set if we are ready to accept that A_0 is equal to 0 in the Poincaré part of the expression. We have redefined

$$i \begin{pmatrix} A_n \\ B_n \end{pmatrix} \rightarrow \begin{pmatrix} A_n \\ B_n \end{pmatrix} \qquad n \neq o$$

With this definition, if we break up system (48) into its real and imaginary parts we obtain two independent systems of equations in the unknowns

$$\left[A_o^R ,\ B_o^I ,\begin{pmatrix} A_n \\ B_n \end{pmatrix}^R ,\begin{pmatrix} C_m \\ D_m \end{pmatrix}^R \right] \quad \text{expressed in terms of} \quad \theta_o^I$$

$$\left[A_o^I ,\ B_o^R ,\begin{pmatrix} A_n \\ B_n \end{pmatrix}^I ,\begin{pmatrix} C_m \\ D_m \end{pmatrix}^I \right] \quad \text{expressed in terms of} \quad \theta_o^R$$

where R and I denote the real or imaginary part of the quantity. Were it not of the assumption of complete reflection at $+ l_3$ all these quantities would have been interrelated.

If θ_o^I or θ_o^R happens to vanish (i. e. if the time origin is suitably chosen) one set of unknowns will vanish as well.

In either set of equations there is $1 \times \infty$ relations between A_o, B_o and A_n, C_m and another infinity between A_o, B_o and B_n, D_m. The two systems of equations would be independent would it not be of the simultaneous presence of A_o and B_o in both sets.

We use this feature by moving B_o to the right and add it to the column vector involving θ_o. In this way we have two systems of equations in the unknowns (A_o, A_n, C_m) and (A_o, B_n, D_m) which we write in matrix notation as

$$\left\{ \begin{array}{l} \Gamma + \times A_+ = \theta_o \delta_+ + B_o \epsilon_+ \\ \Gamma + \times A_- = \theta_o \delta_- + B_o \epsilon_- \end{array} \right\} \qquad (49)$$

where

$$A_+ \equiv \begin{pmatrix} A_o \\ A_n \\ C_m \end{pmatrix} \qquad A_- \equiv \begin{pmatrix} A_o \\ B_n \\ D_m \end{pmatrix}$$

$\delta_{\pm}$, $\epsilon_{\pm}$	are the column vectors of the coefficients of θ_o and B_o in the two systems
$\Gamma_{\pm}$	is the matrix of the coefficients of $A_{\pm}$

The solution of (49) is

$$A_{\pm} = \theta_o \Gamma_{\pm}^{-1} \times \delta_{\pm} + B_o \Gamma_{\pm}^{-1} \times \epsilon_{\pm} \tag{50}$$

and we know

$$(A_+)_o = (A_-)_o$$

so that

$$\theta_o (\Gamma_+^{-1} \times \delta_+)_o + B_o (\Gamma_+^{-1} \times \epsilon_+)_o = \theta_o (\Gamma_-^{-1} \times \delta_-) + B_o (\Gamma_-^{-1} \times \epsilon_-)_o$$

or

$$B_o = \frac{(\Gamma_-^{-1} \times \delta_-)_o - (\Gamma_+^{-1} \times \delta_+)_o}{(\Gamma_+^{-1} \times \epsilon_+)_o - (\Gamma_-^{-1} \times \epsilon_-)_o} \cdot \theta_o \tag{51}$$

The last relation gives an explicit value for B_o which is substituted into the column vectors $\theta_o \delta_{\pm} + B_o \epsilon_{\pm}$

Using $\Gamma_{\pm}^{-1}$ we obtain values for $A_{\pm}$ in (50).

The two systems of equations are quasiindependent because we have assumed the sea to be elongated. We have noted previously that this assumption does not hold well for M_2 and $j = 1$; there, rather than the unknowns C_1 and D_1 we should have written

$$C_1 \pm e^{-2 l_2} D_1 , C_1 e^{-2 l_2} \pm D_1$$

Using the same method as above we express (48) as

$$\Gamma_+ \times A_+ = \theta_0 \delta_+ + B_0 \epsilon_+ + e^{-2l_2} D_1 \zeta_+$$

$$\Gamma_- \times A_- = \theta_0 \delta_- + B_0 \epsilon_- + e^{-2l_2} C_1 \zeta_- \qquad (52)$$

We know

$$(A_\pm)_0 = A_0$$

$$(A_+)_1 = C_1$$

$$(A_-)_1 = D_1$$

which results in the system of three relations between the three unknowns B_0, C_1 and D_1

$$\theta_0(\Gamma_+^{-1} \times \delta_+)_0 + B_0(\Gamma_+^{-1} \times \epsilon_+)_0 + e^{-2l_2}D_1(\Gamma_+^{-1} \times \zeta_+)_0 = \theta_0(\Gamma_-^{-1} \times \delta_-)_0 + B_0(\Gamma_-^{-1} \times \epsilon_-)_0 + e^{-2l_2}C_1(\Gamma_-^{-1} \times \zeta_-)_0$$

$$C_1 = \theta_0 (\Gamma_+^{-1} \times \delta_+)_1 + B_0 (\Gamma_+^{-1} \times \epsilon_+)_1 + e^{-2l_2} D_1 (\Gamma_+^{-1} \times \zeta_+)_1$$

$$D_1 = \theta_0 (\Gamma_-^{-1} \times \delta)_1 + B_0 (\Gamma_-^{-1} \times \epsilon_-)_1 + e^{-2l_2} C_1 (\Gamma_-^{-1} \times \zeta_-)_1 \qquad (53)$$

from which the column vectors can be evaluated and $A \pm$ as well.

Thus by finding the value of the two inverse matrices

$$\Gamma_+^{-1} \quad , \quad \Gamma_-^{-1}$$

we may find four times as many unknowns as the rank of the matrix. $\Gamma_\pm$ for the real and complex parts happen to be identical, and so, if we choose to solve for 40 complex constants in (50), these being determined by two independent sets of 40 linear equations, all we need is to invert two 20 x 20 matrices.

Once these are inverted it is an easy task to find B_0, then the remaining unknowns in 2. From the extra v relations at $-l_1$, and $+l_3$, we get the constants for basins 1 and 3. The number of unknown complex constant determined in this way is 120 which is equivalent to 240 unknown real constants.

Balancing the Kelvin Waves Only

Before we proceed writing out the matrices and the column vectors arising from the previous equations, it is good to pause a moment to verify that the boundary conditions (32) to (37) contain to a zero order of approximation those connecting three channels of constant width and depth.

We have seen previously that the Kelvin waves are the two dimensional equivalent of the sinusoidal waves that appear in a channel of constant width and depth when the motion is purely linear; we now wish to show that boundary conditions (32) to (37) will coalesce into those relating 3 channels of constant width and depth when two dimensional motion is neglected.

The solution is a system of three channels of constant depth h_j, length l_j and width $2a_j$ assuming perfect reflection at the head, is given by

$$v_1 = B_1 \cos K_1(y-l_1) - A_1 \sin K_1(y-l_1) \quad Z_1 = \left(\frac{h_1}{g}\right)^{\frac{1}{2}} \left[-B_1 \sin K_1(y-l_1) - A_1 \cos K_1(y-l_1)\right]$$

$$v_2 = B_2 \cos K_2(y-l_2) - A_2 \sin K_2(y-l_2) \quad Z_2 = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} \left[-B_2 \sin K_2(y-l_2) - A_2 \cos K_2(y-l_2)\right] \qquad (54)$$

$$v_3 = - A_3 \sin K_3(y-l_3) \quad Z_3 = -\left(\frac{h_3}{g}\right)^{\frac{1}{2}} A_3 \cos K_3(y-l_3)$$

The boundary conditions are

$$\begin{aligned} v_1(-l_1) &= \theta_0 \\ h_1a_1v_1(+l_1) &= h_2a_2v_2(-l_2) \\ Z_1(+l_1) &= Z_2(-l_2) \\ h_2a_2v_2(+l_2) &= h_3a_3v_3(-l_3) \\ Z_2(+l_2) &= Z_3(-l_3) \\ v_3(+l_3) &= 0 \qquad (55) \end{aligned} \quad \longleftrightarrow \quad \begin{aligned} & A_1 \sin 2K_1l_1 + B_1 \cos 2K_1l_1 = \theta_0 \\ & h_1a_1B_1 = h_2a_2(A_2 \sin 2K_2l_2 + B_2 \cos 2K_2l_2) \\ & \left(\frac{h_1}{g}\right)^{\frac{1}{2}} A_1 = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} (A_2 \cos 2K_2l_2 - B_2 \sin 2K_2l_2) \\ & h_2a_2 B_2 = h_3a_3 A_3 \sin 2K_3l_3 \\ & \left(\frac{h_2}{g}\right)^{\frac{1}{2}} A_2 = \left(\frac{h_3}{g}\right)^{\frac{1}{2}} A_3 \cos 2K_3l_3 \qquad (56) \end{aligned}$$

If in (42) to (47), we balance the Kelvin waves only we have

$$B_o^{\textcircled{1}} \cos 2K_1l_1 + i\, A_o^{\textcircled{1}} \sin 2K_1l_1 = \frac{\theta_o}{p_{o_1}^{\textcircled{1}}}$$

$$\frac{h_1}{h_2} B_o^{\textcircled{1}} p_{o_1}^{\textcircled{1}} = (B_o^{\textcircled{1}} \cos 2K_2l_2 + i A_o^{\textcircled{2}} \sin 2K_2l_2)\; r_{o_1}^{\textcircled{2}}$$

$$\left(\frac{h_1}{g}\right)^{\frac{1}{2}} A_o^{\textcircled{1}}\, u_{o_2}^{\textcircled{2}} \propto \left(\frac{h_2}{g}\right)^{\frac{1}{2}} \left[A_o^{\textcircled{2}} \cos 2K_2l_2 + i B_o^{\textcircled{2}} \sin 2K_2l_2\right] p_o^{\textcircled{2}} \qquad (57)$$

$$\frac{h_3}{h_2} \left[B_o^{\textcircled{3}} \cos 2K_3l_3 + i A_o^{\textcircled{3}} \sin 2K_3l_3\right] p_o^{\textcircled{3}} = B_o^{\textcircled{2}} r_{o_3}^{\textcircled{2}}$$

$$\left(\frac{h_3}{g}\right)^{\frac{1}{2}} \left[A_o^{\textcircled{3}} \cos 2K_3l_3 + i B_o^{\textcircled{3}} \sin 2K_3l_3\right] u_{o_2}^{\textcircled{3}} \simeq \left(\frac{h_2}{g}\right)^{\frac{1}{2}} A_o^{\textcircled{2}}\, p_o^{\textcircled{2}}$$

$$B_o^{\textcircled{3}} = 0$$

We notice that as $\lambda' \to 0$

$$\frac{1}{p_{o_1}^{\textcircled{1}}} \equiv \frac{\lambda_1 a_1}{\sin h \lambda_1' a_1} \to 1$$

$$\frac{p_o^{\textcircled{2}}}{u_{o_2}^{\textcircled{1}}} \equiv \frac{\sin h\, \lambda_2' a_2}{\lambda_2' a_2} \cdot \frac{\lambda_1' a_2}{\sin h\, \lambda_1' a_2} \to 1 \qquad \frac{p_o^{\textcircled{2}}}{u_{o_2}^{\textcircled{3}}} \equiv \frac{\sin h\, \lambda_2' a_2}{\lambda_2' a_2} \cdot \frac{\lambda_3' a_2}{\sin h\, \lambda_3' a_2} \to 1$$

$$\frac{h_2}{h_1} \cdot \frac{r_{o_1}^{\textcircled{2}}}{p_{o_1}^{\textcircled{1}}} \equiv \frac{h_2}{h_1} \cdot \frac{\sin h \lambda_2' a_2}{\lambda_2' a_1} \cdot \frac{\lambda_1' a_1}{\sin h(\lambda_1' a_1)} \to \frac{h_2}{h_1}\; \frac{a_2}{a_1} \qquad \frac{h_2}{h_3}\; \frac{r_{o_3}^{\textcircled{2}}}{p_o^{\textcircled{3}}} \equiv \frac{h_2}{h_3} \cdot \frac{\sin h\, \lambda_2' a_2}{\lambda_2' a_3} \cdot \frac{\lambda_2' a_3}{\sin h\, \lambda_3' a_3} \to \frac{h_2}{h_3}\; \frac{a_2}{a_3}$$

In this limit (57) becomes

$$\begin{gathered}
iA_o^{\textcircled{1}} \sin 2K_1 l_1 + B_o^{\textcircled{1}} \cos 2K_1 l_1 = \theta_o \\
h_1 a_1 B_o^{\textcircled{1}} = h_2 a_2 [B_o^{\textcircled{2}} \cos 2K_2 l_2 + iA_o^{\textcircled{2}} \sin 2K_2 l_2] \\
\left(\frac{h_1}{g}\right)^{\frac{1}{2}} A_o^{\textcircled{1}} = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} [A_o^{\textcircled{2}} \cos 2K_2 l_2 + iB_o^{\textcircled{2}} \sin 2K_2 l_2] \\
h_2 a_2 B_o^{\textcircled{2}} = h_3 a_3 iA_o^{\textcircled{3}} \sin 2K_3 l_3 \\
\left(\frac{h_2}{g}\right)^{\frac{1}{2}} A_o^{\textcircled{2}} = \left(\frac{h_3}{g}\right)^{\frac{1}{2}} A_o^{\textcircled{3}} \cos 2K_3 l_3
\end{gathered} \tag{58}$$

The two systems of equations (58) and (56) have identical coefficients and the same column vector and therefore we conclude

$$iA_o^{\textcircled{j}} \equiv A_j \qquad B_o^{\textcircled{j}} \equiv B_j$$

i. e. balancing the Kelvin waves only is equivalent to a zero order of approximation to balancing the sinusoidal waves in a system of channels of constant width and depth.

Expressions for the Matrices and the Column Vectors

$$\Gamma_+$$

$$\begin{array}{c|ccc}
 & 0 & \text{odd} & \text{even} \\
\text{even} & \left(\frac{h_2}{g}\right)^{\frac{1}{2}} p_n^{\textcircled{2}} + 2\omega_3\alpha_3 \frac{h_2}{h_3} \cdot \sum_{m_3=1}^{\infty} \frac{P_{m_3 n}}{m} t_{m_3}^{\textcircled{2}}, & -2\omega_2\alpha_2 \frac{P_{mn}}{m} + 2\omega_3\alpha_3 \frac{h_2 a_2}{h_3 a_3} \cdot \sum_{m_3=1}^{\infty} \frac{P_{m_3 n} Q_{m_3 n}}{m_3}, & \frac{\sigma\alpha_2}{K_n} \delta_{nN} + \sigma\alpha_3 \frac{h_2 a_2}{h_3 a_3} \cdot \sum_{m_3=1}^{\infty} \frac{n_3 n\, P_{n_3 N}}{K_{n_3}} \\
\text{odd} & -\sigma\alpha_3 \frac{h_2}{h_3} \sum_{m_3=1}^{\infty} \frac{Q_{m_3 m}}{K_{m_3}} t_{m_3}^{\textcircled{2}}, & -\frac{\sigma\alpha_2}{K_m} \delta_{mM} - \sigma\alpha_3 \frac{h_2 a_2}{h_3 a_3} \cdot \sum_{m_3=1}^{\infty} \frac{Q_{m_3 M} Q_{m_3 m}}{K_{m_3}}, & \frac{2\omega_2\alpha_2 Q_{nm}}{n} - 2\omega_3\alpha_3 \frac{h_2 a_2}{h_3 a_3} \sum_{n_3=2}^{\infty} \frac{P_{n_3 n} Q_{n_3 m}}{n_3}
\end{array} .$$

$$
\Gamma_{-} = \begin{array}{c|ccc}
 & 0 & \text{odd} & \text{even} \\
\hline
\text{even} & \Gamma_{-\,\text{even},0} & 2\omega_2\alpha_2 \dfrac{P_{mn}}{m} - 2\omega_1\alpha_1 \dfrac{h_2a_2}{h_1a_1} \cdot \sum_{m_1=1}^{\infty} \dfrac{P_{m_1n}}{m_1} Q_{m_1m} & \dfrac{\sigma\alpha_2}{K_n} \delta_{nN} + \alpha_1 \dfrac{h_2a_2}{h_1a_1} \cdot \sum_{n_1=2}^{\infty} \dfrac{P_{n_1n}P_{n_1N}}{K_{n_1}} \\
\text{odd} & \Gamma_{-\,\text{odd},0} & -\dfrac{\sigma\alpha_2}{K_m} \delta_{mM} - \sigma\alpha_1 \dfrac{h_2a_2}{h_1a_1} \sum_{m_1=1}^{\infty} \dfrac{Q_{m_1M}Q_{m_1m}}{K_{m_1}} & -2\omega_2\alpha_2 \dfrac{Q_{nm}}{n} + 2\omega_1\alpha_1 \dfrac{h_2a_2}{h_1a_1} \cdot \sum_{n_1=2}^{\infty} \dfrac{P_{n_1n}Q_{n_1m}}{n_1}
\end{array}
$$

$$
\Gamma_{-\,\text{even},0} \equiv \cos 2K_o^{\textcircled{2}} l_2 \left[\left(\frac{h_2}{g}\right)^{\frac{1}{2}} p_n^{\textcircled{2}} + 2\omega_1\alpha_1 \frac{h_2}{h_1} \sum_{m_1=1}^{\infty} \frac{P_{m_1n}}{m_1} t_{m_1}^{\textcircled{2}} \right] - \sin 2K_o^{\textcircled{2}} l_2 \left[\frac{\sigma\alpha_1 h_2}{h_1} \sum_{n_1=2}^{\infty} \frac{P_{n_1n}}{K_{n_1}} r_{n_1}^{\textcircled{2}} - G^{\textcircled{1}} \left\{ \operatorname{cotg} 2K_o^{\textcircled{1}} l_1 \left(\left(\frac{h_1}{g}\right)^{\frac{1}{2}} u_n^{\textcircled{1}} + 2\omega_1\alpha_1 \sum_{m_1=1}^{\infty} \frac{P_{m_1n}}{m_1} q_{m_1}^{\textcircled{1}} \right) + \sigma\alpha_1 \sum_{n_1=2}^{\infty} \frac{P_{n_1n_2}}{K_{n_1}} p_{n_1}^{\textcircled{1}} \right\} \right]
$$

$$
\Gamma_{-\,\text{odd},0} \equiv \sin 2K_o^{\textcircled{2}} l_2 \left[\left(\frac{h_2}{g}\right)^{\frac{1}{2}} q_{m_2}^{\textcircled{2}} + G^{\textcircled{1}} \left\{ -\left(\frac{h_1}{g}\right)^{\frac{1}{2}} v_m^{\textcircled{1}} + \sigma\alpha_1 \operatorname{cotg} 2k_o^{\textcircled{1}} l_{3_1} \cdot \sum_{m_1=1}^{\infty} \frac{Q_{m_1m}}{K_{m_1}} q_{m_1}^{\textcircled{1}} + 2\omega_1\alpha_1 \sum_{n_1=1}^{\infty} \frac{Q_{n_1m}}{n_1} p_{n_1}^{\textcircled{1}} \right\} - 2\omega_1\alpha_1 \frac{h_2}{h_1} \cdot \sum_{n_1=2}^{\infty} \frac{Q_{n_1m}}{n_1} r_{n_1}^{\textcircled{2}} \right] + \cos 2K_o^{\textcircled{2}} l_2 \left[\sigma\alpha_1 \frac{h_2}{h_1} \cdot \sum_{m_1=1}^{\infty} \frac{Q_{m_1m}}{K_{m_1}} \cdot t_{m_1}^{\textcircled{2}} \right]
$$

$$\zeta_+ = \left(\begin{array}{c} -2\omega_2\alpha_2 P_{1n} + 2\omega_3\alpha_3 \dfrac{h_2a_2}{h_3a_3} \displaystyle\sum_{m_3=1}^{\infty} \dfrac{P_{m_3n}\, Q_{m_31}}{m_3} \\ \\ \dfrac{\sigma\alpha_2\delta_{1m}}{K_1} - \sigma\alpha_3 \dfrac{h_2a_2}{h_3a_3} \cdot \displaystyle\sum_{m_3=1}^{\infty} \dfrac{Q_{m_3m}\, Q_{m_31}}{K_{m_3}} \end{array} \right) .$$

$$\zeta_- = \left(\begin{array}{c} 2\omega_2\alpha_2 \cdot P_{1n} - 2\omega_1\alpha_1 \dfrac{h_2a_2}{h_1a_1} \cdot \displaystyle\sum_{m_1=1}^{\infty} \dfrac{P_{m_1n}\, Q_{m_11}}{m_1} \\ \\ \dfrac{\sigma\alpha_2}{K_1}\,\delta_{1n} - \sigma\alpha_1 \dfrac{h_2a_2}{h_1a_1} \cdot \displaystyle\sum_{m_1=1}^{\infty} \dfrac{Q_{m_1m}\, Q_{m_1}}{K_{m_1}} \end{array} \right)$$

$$\delta_+ = 0$$

$$\delta_- = \left(\begin{array}{l} -\dfrac{iF^{\textcircled{1}}}{\sin 2K^{\textcircled{1}}_{o} l_1} \left[\left(\dfrac{h_1}{g}\right)^{\frac{1}{2}} u^{\textcircled{1}}_{n} + 2\omega_1\alpha_1 \displaystyle\sum_{m_1=1}^{\infty} \dfrac{P_{m_1n}}{m_1}\, q^{\textcircled{1}}_{m_1} \right] \\ \\ -\dfrac{i\sigma\alpha_1 F^{\textcircled{1}}}{\sin 2K^{\textcircled{1}}_{o} l_1} \cdot \displaystyle\sum_{m_1=1}^{\infty} \dfrac{Q_{m_1m}}{K_{m_1}}\, q^{\textcircled{1}}_{m_1} \end{array} \right) \begin{array}{l} \text{even} \\ \\ \\ \text{odd} \end{array}$$

$$\epsilon_- = i \begin{pmatrix} \sigma\alpha_3 \dfrac{h_2}{h_3} \cdot \displaystyle\sum_{n_3=2}^{\infty} \dfrac{P_{3n}}{K_{n_3}} r_{n_3}^{\textcircled{2}} - G^{\textcircled{3}} \left[(\mathrm{cotg}\, 2K_{o}^{\textcircled{3}} l_3) \left(\left(\dfrac{h_3}{g}\right)^{\frac{1}{2}} u_n^{\textcircled{3}} + 2\omega_3\alpha_3 \displaystyle\sum_{m_3=1}^{\infty} \dfrac{P_{m_3 n}}{m_3} q_{m_3}^{\textcircled{3}} \right) + \sigma\alpha_3 \displaystyle\sum_{n_3=2}^{\infty} \dfrac{P_{n_3 n}}{K_{n_3}} p_{n_3}^{\textcircled{3}} \right] \\ \\ \left(\dfrac{h_2}{g}\right)^{\frac{1}{2}} q_m^{\textcircled{2}} + G^{\textcircled{3}} \left[-\left(\dfrac{h_3}{g}\right)^{\frac{1}{2}} v_m^{\textcircled{3}} + \sigma\alpha_3\, \mathrm{cotg}\, 2K_{o}^{\textcircled{3}} l_3 \cdot \displaystyle\sum_{m_3=1}^{\infty} \dfrac{Q_{m_3 m}}{K_{m_3}} q_{m_3}^{\textcircled{3}} + 2\omega_3\alpha_3 \displaystyle\sum_{n_3=2}^{\infty} \dfrac{Q_{n_3 m}}{n_3} p_{n_3}^{\textcircled{3}} \right] - 2\omega_3\alpha_3 \dfrac{h_2}{h_3} \displaystyle\sum_{n_3=2}^{\infty} \dfrac{Q_{n_3 m}}{n_3} r_{n_3}^{\textcircled{2}} \end{pmatrix} \begin{array}{l} \textbf{even} \\ \\ \\ \textbf{odd} \end{array}$$

$$\epsilon_- = i \begin{bmatrix} \begin{array}{l} -\cos 2K_{o}^{\textcircled{2}} l_2 \left[\sigma\alpha_1 \dfrac{h_2}{h_1} \cdot \displaystyle\sum_{n_1=2}^{\infty} \dfrac{P_{n_1 n}}{K_{n_1}} r_{n_1}^{\textcircled{2}} - G^{\textcircled{1}} \left\{ \mathrm{cotg}\, 2K_{o}^{\textcircled{1}} l_1 \left(\dfrac{h_1}{g}\right)^{\frac{1}{2}} u_n^{\textcircled{1}} + 2\omega_1\alpha_1 \displaystyle\sum_{m_1=1}^{\infty} \dfrac{P_{m_1 n}}{m_1} q_{m_1}^{\textcircled{1}} + \sigma\alpha_1 \displaystyle\sum_{n_1=2}^{\infty} \dfrac{P_{n_1 n}}{K_{n_1}} p_{n_1}^{\textcircled{1}} \right\} \right] \\ - \sin 2K_{o}^{\textcircled{2}} l_2 \left[\left(\dfrac{h_2}{g}\right)^{\frac{1}{2}} p_n^{\textcircled{2}} + 2\omega_1\alpha_1 \dfrac{h_2}{h_1} \displaystyle\sum_{m_1=1}^{\infty} \dfrac{P_{m_1 n}}{m_1} t_{m_1}^{\textcircled{2}} \right] \end{array} \\ \\ \\ \begin{array}{l} \cos 2K_{o}^{\textcircled{2}} l_2 \left[\left(\dfrac{h_2}{g}\right)^{\frac{1}{2}} q_m^{\textcircled{2}} + G^{\textcircled{1}} \left\{ -\left(\dfrac{h_1}{g}\right)^{\frac{1}{2}} v_m^{\textcircled{1}} + \sigma\alpha_1\, \mathrm{cotg}\, 2K_{o}^{\textcircled{1}} l_1 \displaystyle\sum_{m_1=1}^{\infty} \dfrac{Q_{m_1 m}}{K_{m_1}} q_{m_1}^{\textcircled{1}} + 2\omega_1\alpha_1 \displaystyle\sum_{n_1=2}^{\infty} \dfrac{Q_{n_1 m}}{n_1} p_{n_1}^{\textcircled{1}} \right\} - 2\omega_1\alpha_1 \dfrac{h_2}{h_1} \displaystyle\sum_{n_1=2}^{\infty} \dfrac{Q_{n_1 m}}{n_1} r_{n_1}^{\textcircled{2}} \right] \\ - \sin 2K_{o}^{\textcircled{2}} l_2 \cdot \sigma\alpha_1 \dfrac{h_2}{h_1} \displaystyle\sum_{m_1=1}^{\infty} \dfrac{Q_{m_1 m}}{K_m} t_{m_1}^{\textcircled{2}} \end{array} \end{bmatrix} \begin{array}{l} \textbf{even} \\ \\ \\ \\ \\ \textbf{odd} \end{array}$$

The elements of the matrices and of the column vectors involve a set of infinite series which have first to be evaluated before we can proceed to the solution of the linear equations. Such infinite series are given by the formulas

$$\sum_{n_k=2}^{\infty} \frac{P_{n_k n}}{K_{n_k}} \; p_{n_k}^{\textcircled{k}} = \frac{32}{\pi^3} \; \frac{a_k}{a_2} \cdot (\lambda'_k a_k) \cdot \sinh(\lambda'_k a_k) \, (-)^{\frac{n}{2}} \sum_{n_k=2}^{\infty} \frac{n_k}{K_{n_k}} \; \frac{(-)^{\frac{n_k}{2}} \sin(n'_k a_2)}{\left(n_k^2 + \lambda_k^2\right)\left(n_k^2 - \left(\frac{n a_k}{a_2}\right)^2\right)} \qquad \textcircled{2}$$

$$\sum_{m_k=1}^{\infty} \frac{P_{m_k n}}{m_k} \; q_{m_k}^{\textcircled{k}} \equiv \frac{32}{\pi^3} \cdot (\lambda' a_k) \cdot \cosh(\lambda'_k a_k) \, (-)^{\frac{n}{2}} \sum_{m_k=1}^{\infty} (-)^{\frac{m_k-1}{2}} \frac{\sin(m'_k a_2)}{\left(m_k^2 + \lambda_k^2\right)\left(m_k^2 - \left(\frac{n a_k}{a_2}\right)^2\right)} \qquad \textcircled{1}$$

$$\sum_{n_k=2}^{\infty} \frac{Q_{n_k m}}{n_k} \; p_{n_k}^{\textcircled{k}} \equiv - \frac{32}{\pi^3} \; \frac{a_k}{a_2} \cdot (\lambda'_k a_k) \cdot \sinh(\lambda'_2 a_k) \, (-)^{\frac{m-1}{2}} \sum_{n_k=2}^{\infty} \frac{(-)^{\frac{n_k}{2}} \cos(n'_k a_2)}{\left(n_k^2 + \lambda_k^2\right)\left(n_k^2 - \left(\frac{m a_k}{a_2}\right)^2\right)} \qquad \textcircled{12}$$

$$\sum_{m_k=1}^{\infty} \frac{Q_{m_k m}}{K_{m_k}} \; q_{m_k}^{\textcircled{k}} \equiv - \frac{32}{\pi^3} \; \frac{a_k}{a_2} \cdot (\lambda'_k a_k) \cosh(\lambda'_k a_k) \, (-)^{\frac{m-1}{2}} \sum_{m_k=1}^{\infty} \frac{m_k}{K_{m_n}} \; \frac{(-)^{\frac{m_k-1}{2}} \cos(m'_k a_2)}{\left(m_k^2 + \lambda_k^2\right)\left(m_k^2 - \left(\frac{m a_k}{a_2}\right)^2\right)} \qquad \textcircled{11}$$

$$\sum_{n_k=2}^{\infty} \frac{P_{n_k n}}{K_{n_k}} \; r_{n_k}^{\textcircled{2}} \equiv \frac{16}{\pi^2} \; \frac{a_k}{a_2} \, (-)^{\frac{n}{2}} \left\{ \frac{\lambda_2 a_k}{a_2} \; \sinh \lambda'_2 a_2 \cdot \sum_{n_k=2}^{\infty} \frac{n_k}{K_{n_k}} \cdot \frac{\sin(n'_k a_2) \cos(n'_k a_2)}{\left(n_k^2 + \left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(n_k^2 - \left(\frac{n a_k}{a_2}\right)^2\right)} \right. \qquad \textcircled{4}$$

$$\left. + \cosh \lambda'_2 a_2 \sum_{n_k=2}^{\infty} \frac{n_k^2}{K_{n_k}} \cdot \frac{\sin^2(n'_k a_2)}{\left(n_k^2 + \left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(n_k^2 - \left(\frac{n a_k}{a_2}\right)^2\right)} \right\} \qquad \textcircled{6}$$

$$\sum_{m_k=1}^{\infty} \frac{P_{m_k n}}{m_k} \; t_{m_k}^{\textcircled{2}} \equiv \frac{16}{\pi^2} \; \frac{a_k}{a_2} \, (-)^{\frac{n}{2}} \left\{ \frac{\lambda_2 a_k}{a_2} \; \cosh \lambda'_2 a_2 \sum_{m_k=1}^{\infty} \frac{\sin^2(m'_k a_2)}{\left(m_k^2 + \left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(m_k^2 - \left(\frac{n a_k}{a_2}\right)^2\right)} \right. \qquad \textcircled{5}$$

$$\left. - \sinh \lambda'_2 a_2 \sum_{m_k=2}^{\infty} \frac{m_k \sin(m'_k a_2) \cos(m'_k a_2)}{\left(m_k^2 + \left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(m_k^2 - \left(\frac{n a_k}{a_2}\right)^2\right)} \right\} \qquad \textcircled{3}$$

$$\sum_{m_k=1}^{\infty} \frac{Q_{m_k m}}{K_{m_k}} \overset{\textcircled{2}}{t}_{m_k} \equiv -\frac{16}{\pi^2} \frac{a_k}{a_2} (-)^{\frac{m-1}{2}} \left\{ \frac{\lambda_2 a_k}{a_2} \cosh \lambda_2' a_2 \sum_{m_k=1}^{\infty} \frac{m_k}{K_{m_k}} \cdot \frac{\sin(m_k' a_2) \cos(m_k' a_2)}{\left(m_k^2 + \left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(m_k^2 - \left(\frac{m a_k}{a_2}\right)^2\right)} \right.$$ ⑥

$$\left. - \sinh \lambda_2' a_2 \sum_{m_k=1}^{\infty} \frac{m_k^2}{K_{m_k}} \cdot \frac{\cos^2(m_k' a_2)}{\left(m_k^2 + \left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(m_k^2 - \left(\frac{m a_k}{a_2}\right)^2\right)} \right\}$$ ⑦

$$\sum_{n_k=2}^{\infty} \frac{Q_{n_k m}}{n_k} \cdot \overset{\textcircled{1}}{\tau}_{n_k} \equiv -\frac{16}{\pi^2} \cdot \frac{a_k}{a_2} (-)^{\frac{m-1}{2}} \left\{ \frac{\lambda_2 a_k}{a_2} \sinh \lambda_2' a_2 \sum_{n_k=2}^{\infty} \frac{\cos^2(m_k' a_2)}{\left(n_k^2 + \left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(n_k^2 - \left(\frac{m a_k}{a_2}\right)^2\right)} \right.$$ ⑨

$$\left. + \cosh \lambda_2' a_2 \sum_{n_k=2}^{\infty} \frac{n_k \sin(n_k' a_2) \cos(n_k' a_2)}{\left(n_k^2 + \left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(n_k^2 - \left(\frac{m a_k}{a_2}\right)^2\right)} \right\}$$ ⑩

$$\sum_{n_k=2}^{\infty} \frac{P_{n_k n} P_{n_k N}}{K_{n_k}} \equiv \frac{16}{\pi^2} \left(\frac{a_k}{a_2}\right)^2 (-)^{\frac{n+N}{2}} \sum_{n_k=2}^{\infty} \frac{n_k^2}{K_{n_k}} \cdot \frac{\sin^2(n_k' a_2)}{\left(n_k^2 - \left(\frac{n a_k}{a_2}\right)^2\right)\left(n_k^2 - \left(\frac{N a_k}{a_2}\right)^2\right)}$$

$$\sum_{m_k=1}^{\infty} \frac{P_{m_k n} Q_{m_k m}}{m_k} \equiv -\frac{16}{\pi^2} \left(\frac{a_k}{a_2}\right)^2 (-)^{\frac{n+m-1}{2}} \sum_{m_k=1}^{\infty} \frac{m_k \cdot \sin(m_k' a_2) \cos(m_k' a_2)}{\left(m_k^2 - \left(\frac{n a_k}{a_2}\right)^2\right)\left(m_k^2 - \left(\frac{m a_k}{a_2}\right)^2\right)}$$

$$\sum_{n_k=2}^{\infty} \frac{P_{n_k n} Q_{n_k m}}{n_k} \equiv -\frac{16}{\pi^2} \left(\frac{a_k}{a_2}\right)^2 (-)^{\frac{n+m-1}{2}} \sum_{n_k=2}^{\infty} \frac{n_k \cdot \sin(n_k' a_2) \cos(n_k' a_2)}{\left(m_k^2 - \left(\frac{n a_k}{a_2}\right)^2\right)\left(n_k^2 - \left(\frac{m a_k}{a_2}\right)^2\right)}$$

$$\sum_{m_k=1}^{\infty} \frac{Q_{m_k M} Q_{m_k m}}{K_{m_k}} \equiv -\frac{16}{\pi^2} \left(\frac{a_k}{a_2}\right)^2 (-)^{\frac{m+M}{2}} \sum_{m_k=1}^{\infty} \frac{m_k^2}{K_{m_k}} \cdot \frac{\cos^2(m_k' a_2)}{\left(m_k^2 - \left(\frac{m a_k}{a_2}\right)^2\right)\left(m_k^2 - \left(\frac{M a_k}{a_2}\right)^2\right)}$$

Tables 5 and 6 give the numerical value of the part of the infinite series which is contained within the summation sign.

The first set of twelve summations resulting from the interaction of the Kelvin and Poincaré waves are labelled by circled numbers, the order of which will become obvious.

The latter set of four summations resulting from the interaction of the Poincaré waves depends on the values of the integers n, N, m and M. In Table 6 they are therefore to be recognized by the value of ${}_jJ$ inserted after the value of a given summation.

Table V

Table V gives the values of the summations arising from the interaction between the Kelvin and the Poincaré waves.

These summations are evaluated sequentially for $j = 0, 1, \ldots 19$ in an order which is convenient for automatic computation.

The even lines give summations ① to ⑥ while the odd lines give ⑦ to ⑫.

There are four sets of such quantities; two for M_2, two for K_1. The first of the pair gives the summations for $k = 1$, while the other gives the summations for $k = 3$.

00	-.61631	.05970	-.35169	-.11938	-.36906	.00577
01	.09544	.00100	-.11646	.08470	-.17994	.03827
02	.02870	.01268	.02491	.02817	.00580	.00384
03	.01446	.00216	.00978	.00994	.01787	.00200
04	.00719	.00297	.00630	.00829	.00146	.00037
05	.00581	.00085	.00329	.00359	.00635	.00065
06	.00319	.00130	.00280	.00402	.00065	.00007
07	.00318	.00044	.00163	.00184	.00323	.00032
08	.00180	.00072	.00158	.00241	.00037	.00001
09	.00203	.00027	.00097	.00111	.00195	.00019
10	.00115	.00046	.00101	.00163	.00024	.00000
11	.00142	.00018	.00064	.00075	.00130	.00012
12	.00080	.00032	.00070	.00120	.00017	.00000
13	.00107	.00013	.00045	.00053	.00093	.00009
14	.00058	.00023	.00052	.00095	.00013	.00000
15	.00086	.00010	.00034	.00042	.00070	.00006
16	.00045	.00017	.00040	.00087	.00012	.00000
17	.00079	.00008	.00025	.00001	.00054	.00005
18	.00035	.00014	.00031	.00058	.00020	.00001
19	.00055	.00006	.00020	.00000	.00043	.00004
00	-.91721	.03979	-.83379	.05402	-.27682	.02957
01	-.12644	.06761	-.47669	.09657	-.51729	.08329
02	-.15329	.05865	-.14607	.05282	.00436	.05147
03	.02174	.01279	.03136	.01134	.03623	.01129
04	.03855	.01108	.03695	.01623	.00110	.00867
05	.00934	.00522	.00983	.00410	.01189	.00467
06	.01715	.00451	.01646	.00809	.00049	.00326
07	.00531	.00275	.00471	.00210	.00587	.00247
08	.00965	.00244	.00926	.00492	.00028	.00168
09	.00347	.00169	.00275	.00127	.00349	.00151

10	.00617	.00153	.00593	.00335	.00018	.00101
11	.00247	.00113	.00180	.00085	.00232	.00102
12	.00429	.00105	.00412	.00244	.00013	.00067
13	.00187	.00081	.00127	.00061	.00165	.00073
14	.00315	.00076	.00302	.00188	.00010	.00048
15	.00147	.00061	.00094	.00046	.00123	.00055
16	.00241	.00058	.00232	.00150	.00008	.00036
17	.00119	.00048	.00073	.00036	.00096	.00043
18	.00190	.00045	.00183	.00123	.00006	.00028
19	.00100	.00038	.00058	.00029	.00076	.00034
00	-.32586	.03769	-.10939	.06599	-.26898	.02897
01	.03863	.00820	.02099	.03113	.04592	.01056
02	.00358	.00165	.00341	.00669	.00106	.00091
03	.00525	.00107	.00190	.00351	.00497	.00103
04	.00089	.00040	.00085	.00201	.00026	.00029
05	.00207	.00039	.00065	.00127	.00178	.00036
06	.00039	.00017	.00038	.00102	.00011	.00014
07	.00114	.00020	.00033	.00067	.00091	.00018
08	.00022	.00009	.00022	.00073	.00001	.00008
09	.00077	.00012	.00019	.00009	.00054	.00011
10	.00014	.00006	.00013	.00039	.00016	.00005
11	.00047	.00007	.00012	.00007	.00036	.00007
12	.00009	.00004	.00009	.00026	.00011	.00004
13	.00033	.00005	.00009	.00005	.00026	.00005
14	.00007	.00003	.00006	.00018	.00008	.00003
15	.00024	.00004	.00006	.00004	.00019	.00004
16	.00005	.00002	.00005	.00014	.00006	.00002
17	.00019	.00003	.00005	.00003	.00015	.00003
18	.00004	.00001	.00004	.00011	.00005	.00001
19	.00015	.00002	.00004	.00002	.00012	.00002

00	-.55315	.05642	-.33037	-.11771	-.42375	.01591
01	.09908	.00515	.09599	.07454	-.13860	.02993
02	.01908	.00844	.01882	.02252	.00168	.00106
03	.01369	.00188	.00831	.00840	.01417	.00204
04	.00477	.00201	.00472	.00654	.00042	.00011
05	.00538	.00071	.00284	.00303	.00506	.00069
06	.00212	.00088	.00210	.00317	.00018	.00010
07	.00291	.00037	.00141	.00155	.00257	.00035
08	.00119	.00049	.00118	.00190	.00010	.00007
09	.00185	.00022	.00084	.00094	.00155	.00021
10	.00076	.00031	.00075	.00129	.00006	.00005
11	.00130	.00015	.00056	.00064	.00104	.00014
12	.00053	.00022	.00052	.00097	.00004	.00004
13	.00099	.00011	.00040	.00048	.00074	.00010
14	.00038	.00016	.00039	.00086	.00002	.00003
15	.00084	.00008	.00029	.00001	.00056	.00007
16	.00029	.00012	.00029	.00056	.00022	.00003
17	.00059	.00006	.00022	.00002	.00043	.00005
18	.00023	.00009	.00023	.00042	.00017	.00002
19	.00045	.00005	.00018	.00002	.00034	.00004

Table VI

Table VI gives the numerical value of the summation arising from the interaction of the Poincaré waves. There are 20^2 such summations if we choose to terminate for $j = 19$.

They are conveniently labelled in terms of j and J which follow these quantities and on which they depend. They are given in the same order as the previous summations. It should be noted that such a type of interaction is almost negligible.

```
.00772 00 00   -.13719 00 01   .00213 00 02   .01216 00 03

.00044 00 04    .00431 00 05   .00019 00 06   .00219 00 07

.00010 00 08    .00132 00 09   .00006 00 10   .00088 00 11

.00004 00 12    .00063 00 13   .00003 00 14   .00047 00 15

.00002 00 16    .00036 00 17   .00002 00 18   .00029 00 19

.00546 01 00    .04274 01 01   .00161 01 02   .00365 01 03

.00031 01 04    .00130 01 05   .00013 01 06   .00066 01 07

.00007 01 08    .00039 01 09   .00004 01 10   .00026 01 11

.00003 01 12    .00019 01 13   .00002 01 14   .00014 01 15

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.00000 14 16    .00000 14 17    .00000 14 18    .00000 14 19
.00000 15 00    .00008 15 01    .00000 15 02    .00000 15 03
.00000 15 04    .00000 15 05    .00000 15 06    .00000 15 07
.00000 15 08    .00000 15 09    .00000 15 10    .00000 15 11
.00000 15 12    .00000 15 13    .00000 15 14    .00000 15 15
.00000 15 16    .00000 15 17    .00000 15 18    .00000 15 19
.00001 16 00    .00006 16 01    .00000 16 02    .00000 16 03
.00000 16 04    .00000 16 05    .00000 16 06    .00000 16 07
.00000 16 08    .00000 16 09    .00000 16 10    .00000 16 11
.00000 16 12    .00000 16 13    .00000 16 14    .00000 16 15
.00000 16 16    .00000 16 17    .00000 16 18    .00000 16 19
.00000 17 00    .00006 17 01    .00000 17 02    .00000 17 03
.00000 17 04    .00000 17 05    .00000 17 06    .00000 17 07
.00000 17 08    .00000 17 09    .00000 17 10    .00000 17 11
.00000 17 12    .00000 17 13    .00000 17 14    .00000 17 15
.00000 17 16    .00000 17 17    .00000 17 18    .00000 17 19
```

```
.00001 18 00   .00004 18 01   .00000 18 02   .00000 18 03
.00000 18 04   .00000 18 05   .00000 18 06   .00000 18 07
.00000 18 08   .00000 18 09   .00000 18 10   .00000 18 11
.00000 18 12   .00000 18 13   .00000 18 14   .00000 18 15
.00000 18 16   .00000 18 17   .00000 18 18   .00000 18 19
.00000 19 00   .00005 19 01   .00000 19 02   .00000 19 03
.00000 19 04   .00000 19 05   .00000 19 06   .00000 19 07
.00000 19 08   .00000 19 09   .00000 19 10   .00000 19 11
.00000 19 12   .00000 19 13   .00000 19 14   .00000 19 15
.00000 19 16   .00000 19 17   .00000 19 18   .00000 19 19
```

The Boundary Condition on v at the Mouth

We have not yet specified the type of motion we wish to assume at the mouth. There is no observational material from which we can draw and we will therefore use the simplest boundary condition we can think of:

$$\theta_o = -i \qquad (59)$$

$$B_n^{①} = D_m^{①} = 0$$

i. e. we assume the velocity at the mouth to be of a purely Kelvin type with the maximum inward current running approximately $\frac{1}{4}$ period before high water.

Once the calculations will be concluded we will find out that this boundary condition leads to a cotidal chart for M_2 which is in complete qualitative agreement with the chart shown in Fig. 2 based exclusively on coastal observations.

Our assumption therefore must have some physical justification.

Table VII

Numerical Values of the Matrices and of the Column Vectors

All the matrices and column vectors have been multiplied by the factor 10^4; the exceptions are the inverse matrices which have their natural values.

The values for M_2 at the lower and upper boundaries are given first, then those for K_1.

All the matrices and the column vectors have been renormalized so that the diagonal elements of the Gamma matrices have absolute value 1.

```
                             GAMMA-   M2
  1      10000   1369     10-   159      2     57      1-    29      1     18
              -    12             8      -      6             5             4
  2       4501- 10000-  3741     60    749-    21-   320     11    178-     7
       -   113      5     78-     3-    57      2     44-     2-    35      2
  3       2179-  7626- 10000-  4917      4   1173-     1-   548      1    320
       -     1-   211           150      -    112            87      -     68
  4       1537-   193   7312  10000   5218-     7-  1354      4    666-     2
       -   402      2    271      2-   195           148      -    117    749
  5       1372-  3124-     7   6953  10000   5418      -   1480
              -   465           319      -    232           178      -    141
  6        898-   115   2906     12-  6781- 10000-  5549      3   1566
       -   814           514      -    356           263      -    204
  7        784-  2010-     4   2705      -   6659- 10000-  5637           1628
              -   862           550      -    386           290      -    227
  8        633-    82   1899      9-  2592          6578  10000   5700
       -  1677           902      -    581           412           312
  9        584-  1491          1773      -   2506          6513  10000   5745
              -  1714           931      -    604           436      -    327
 10        479-    61   1427      6-  1689          2441      -   6466- 10000
       -  5786          1745      -    953           630      -    452
 11        477-  1183-     6   1337      -   1628          2396      -   6427
       - 10000-  5814          1771      -    978           644      -    471
 12        395-    54   1151      7-  1287          1580      -   2357
          6396  10000   5838      -   1792           995      -    661
 13        394-   981          1085      -   1234          1539      -   2327
                 6372  10000   5866      -   1807          1011      -    677
 14        322-    40    966      8-  1039          1192      -   1514
          2303      -   6353- 10000-  5878          1828      -   1023
 15        364-   841           911      -    997          1162      -   1483
                 2281      -   6331- 10000-  5898          1839      -   1041
 16        269-    37    826      -    873           966      -   1133
          1467      -   2256          6314  10000   5905      -   1848
 17        327-   733           793      -    842           942      -   1120
                 1447      -   2250          6303  10000   5927      -   1863
 18        284-    32    726      -    758           821      -    926
          1095      -   1432          2232      -   6295- 10000-  5926
 19        312-   658           702      -    736           803      -    903
                 1081      -   1416          2218      -   6276- 10000-  5942
 20        247-    35    647      -    671           718      -    776
           894      -   1071          1400      -   2212          6271  10000
```

```
                             GAMMA+   M2
 1       10000-   753     13    135-     2-    49      1     25       -    15
                   10       -     7             6       -     4             3
 2     -   932- 10000   2943    107-   590-    37    253     19-   140-    11
            89      8-    62-     5     46      4-    35-     3     27      3
 3        1251   5650- 10000   4856     17-  1169-     7    546      4-   319
       -     2    211      2-   149-     1    110      1-    86            69
 4     -    61-   425-  7186  10000-  5211-            5      6    665
           401      3-   270-     2    194      1-   148      1    116
 5         646   2229-    35-  6911  10000-  5432-     2   1487      1-   754
                  467       -   322           235       -   178           141
 6          28-   251-  2861     19   6768- 10000   5552      3-  1574      2
           820      1-   517           356       -   263           204
 7         431   1423-    22-  2678      4   6649- 10000   5639       -  1631
                  866       -   550           386       -   290           227
 8          48    178-  1867     13   2561-     4-  6573  10000-  5693
          1677       -   901           581       -   412           312
 9         322   1055-    15-  1753      5   2496       -  6508  10000-  5745
                 1714       -   931           604       -   436           327
10          45-   139-  1405     11   1683-     6-  2441          6466- 10000
          5786       -  1745           959       -   630           452
11         254    836-    12-  1325          1622       -  2396          6427
       - 10000   5814       -  1771           978       -   644           471
12          48-   109-  1131      7   1281       -  1580          2357
       -  6396  10000-  5845          1792       -   995           661
13         203    698-     7-  1070          1226       -  1545          2325
              -  6367  10000-  5862          1805       -  1010           676
14          48-    97-   950      8   1039       -  1192          1514
       -  2303          6353- 10000   5878       -  1828          1023
15         182    590-     9-   893           997       -  1162          1483
              -  2272          6331- 10000   5898       -  1839          1041
16          46-    84-   743      9    873       -   966          1142
       -  1467          2256       -  6314  10000-  5905          1848
17         159    515-    10-   783           842       -   942          1120
              -  1447          2250       -  6303  10000-  5927          1863
18          53-    74-   716           758       -   821           926
       -  1095          1432       -  2232          6284- 10000   5926
19         145    457-    11-   691           736       -   803           903
              -  1081          1416       -  2218          6276- 10000   5942
20          47-    59-   635           671       -   694           776
       -   894          1071       -  1412          2212       -  6271  10000
```

```
                                 GAMMA -   M2
  1        .9355  .2190 -.1142 -.0336 -.0174 -.0235  .0040 -.0114  .0122  .0085
          -.0062 -.0048  .0001 -.0039  .0023 -.0026  .0039  .0004  .0028  .0039
  2        .4472-2.0743 1.6087 1.0062 -.1849  .2108 -.1162  .0364  .0340  .1723
          -.1303 -.0020 -.0272  .0226 -.0013  .0689 -.0198  .1644 -.2839 -.2488
  3       -.0467 3.4333-5.3387-4.0935 1.8701  .1295  .3603  .4650 -.9345-1.3755
           .9469  .2358  .0752  .0209 -.1222 -.2810 -.0883 -.9310 1.4032 1.1279
  4       -.1319-3.3032 6.1458 7.5926-5.5187-2.9007  .3862-1.8970 3.6171 4.0077
         -2.3288 -.6413 -.0171 -.0787  .4421  .6338  .4123 2.2487-3.1070-2.3111
  5       -.0306  .8117-3.1781-6.3190 9.0398 7.8858-3.6014 2.5607-6.8911-7.1876
          3.6066  .6419  .2092  .0678 -.3945 -.6339 -.4557-2.3354 3.1459 2.1992
  6        .2471 1.4030-1.2803 1.8682-7.3423 -11.2828.1450  .6202 5.9593 7.2183
         -3.3474  .1805 -.8748  .1009 -.4823  .0717 -.4104 -.0175 -.4213 -.4667
  7       -.2050-1.1028 2.5053 1.9619 2.1784 7.8335 -10.82-7.0028  .5583-3.6900
          2.8222 -.4518 1.9214  .3233 1.6947 1.1795 1.6514 3.9924-4.0706-1.9978
  8       -.0765 -.6555  .1387-1.4681 2.1097 -.9874 7.1472 11.79 -9.0769-2.6714
         -1.7052 -.7491-1.7762-1.1374-1.7535-2.0085-1.9837-6.2244 6.8310 3.3125
  9        .2227 1.1646-2.2517-1.2538-1.9236-3.2221-1.1236-8.6574 12.72  8.3885
         -1.5265 1.5833 -.3426  .8451  .2382 1.0282  .8843 3.7361-4.5138-2.1459
 10       -.0694  .0308 1.0032 1.8493 -.7255 1.8378-2.3665 1.5936-7.5268 -10.44
          6.6784 1.1644 1.7278 1.0357  .5430  .8012 -.1181  .2132  .0323  .0435
 11       -.1274 -.9059 1.3093  .2448 1.5236 1.3899 1.2704 2.8849  .3489 6.7020
                 -7.0019  .9948-1.9853  .7304-1.0456  .4252-1.4337 2.1142 1.0308
 12        .0877  .2548-1.2261-1.5700  .1773-1.6154 1.0932-1.6385 2.5105-1.6177
          8.0339 11.14 -7.0158-1.1735-1.7405 -.9928 -.4247 -.4492 -.2882 -.3095
 13        .0923  .6764 -.6686  .3306-1.3183 -.6624-1.1147-1.4114 -.5440-1.4792
         -1.8725-7.9091  11.23 7.2837-1.1105 2.1051 -.9456 1.3869-1.9475 -.8680
 14       -.1298 -.4555 1.2592 1.2035  .3035 1.5508 -.5498 1.5767-1.4999 1.2685
         -2.5220 1.4305-7.9910 -11.22 7.0549 1.1815 1.6128  .7043  .8186  .7228
 15       -.0131 -.3783  .0896 -.7183  .9813  .0075 1.0475  .5798  .7202  .3329
          1.8711 2.4930 1.4992 7.7781 -11.10-7.3245 1.3688-2.2340 2.1624  .6902
 16        .1198  .4808-1.0473 -.7152 -.6221-1.3072  .1161-1.4401  .9631 -.9000
           .9980-1.4709 2.4957-1.4586 7.8901 11.03 -6.8911-1.0286-2.0124-1.3706
 17       -.0623  .1574  .2931  .8331 -.5323  .5338 -.9040  .1010 -.9140  .1244
         -1.7091-1.0720-1.6013-2.3106-1.6721-7.5674 10.52  7.3045-2.7399 -.1487
 18       -.0441 -.4966  .8153  .3108  .7239  .8776  .2692 1.1431 -.4315  .6043
          -.0119 1.4857-1.0087 1.6031-2.3810 1.6639-7.8355 -10.98 7.6522 2.7550
 19        .0673  .2011 -.8127 -.9473 -.0110-1.0346  .5868 -.8326 1.0230 -.5024
          1.3137 -.1754 1.7402  .4581 2.2518 1.8189 3.0042 7.7903-8.4141-3.9138
 20       -.0214  .1043  .1592  .4994 -.3459  .2802 -.5234  .0273 -.4889  .0614
          -.8548 -.5139 -.6970 -.8840 -.5347-1.4991  .0380-2.7726 4.0313 3.1902
```

```
                                    GAMMA +  M2
  1        .9947 -.0859 -.0125 -.0262 -.0395  .0353  .0192  .0079  .0286 -.0262
          -.0095 -.0052 -.0063 -.0007 -.0038  .0003 -.0030  .0017 -.0009  .0013
  2       -.0420-1.4059 -.6991  .4663  .3816 -.4735 -.4277 -.0354 -.5305  .5413
           .1784  .1450  .1361  .0438  .1009 -.0048  .0655 -.0258  .0343 -.0338
  3        .1598-1.3766-2.6851 2.6894 3.3318-3.5660-2.3838 -.6677-3.1845 2.8886
           .8800  .7243  .6479  .2187  .4806 -.0264  .3007 -.1182  .1564 -.1541
  4        .0863 -.8553-2.4264 6.0229 9.1430 -10.08-6.0655-2.0997-7.7458 6.3320
          1.5452 1.6664 1.1766  .6069  .9219  .0743  .6083 -.1361  .3820 -.3095
  5       -.0655  .4296 -.9219 6.0101 13.01  -14.38-8.4877-2.3009-8.8731 6.5983
           .9550 2.0341  .9103  .9384  .8447  .3342  .6423  .0543  .5320 -.3245
  6       -.1061 1.2594 1.3180 2.6583 9.4825 -13.76-9.6760  .9908-5.0225 4.1335
           .1797 1.7171  .5027  .9296  .6096  .4308  .5470  .1463  .5188 -.2753
  7       -.0100  .8082 1.6773-1.3272 1.8263-6.3599-9.1725 6.9785 2.9133  .1777
           .5575  .2657  .4698  .0470  .3347 -.0679  .2103 -.1219  .0890 -.1095
  8        .0744 -.0211  .1567-2.0579-3.1787  .3888-5.7317 11.07  9.4624-3.5835
          1.1467-1.6374  .2417-1.0744 -.1856 -.6499 -.3479 -.3526 -.4834  .1655
  9        .0413 -.1483 -.9394  .1150-1.8917 2.0797-1.9308 7.8282 10.78 -6.8846
         -1.1473-1.8144 -.9216 -.7994 -.8232 -.2394 -.6100  .0317 -.4399  .3017
 10       -.0391  .3281 -.2853 1.4507 1.5549  .0172  .9254 1.3190 6.0346-9.4213
         -6.6792 1.6415-1.5405 1.1451 -.5335  .9059  .0224  .5809  .3592  .0182
 11       -.0463  .3916  .7303  .2850 1.7070-1.1909 1.4301-2.5715 1.1263-7.5608
                 7.2605 1.1987 1.9912 1.0356  .8572  .9583  .1782  .7492 -.3881
 12        .0184 -.1134  .4808-1.1166 -.8259 -.0585  .1342-1.3288-1.0648-2.7178
         -8.1829  10.80 6.9089-1.3174 1.6888-1.0520  .5636 -.8695 -.1705 -.2209
 13        .0525 -.3724 -.4631 -.5815-1.6482  .9756 -.9056 1.3454 -.6514 1.8390
         -1.4832 7.5851 11.09 -7.3754-1.3507-1.9986-1.2591 -.6856-1.1847  .4675
 14        .0111 -.0067 -.5312  .7380  .1709  .2758 -.4943 1.3710  .2881 2.3093
          2.7759 1.6361 8.0856 -11.14-7.0601 1.2506-1.7703 1.1572 -.3160  .5783
 15       -.0373  .3773  .3182  .6545 1.3722 -.7203  .4881 -.5614  .4407 -.3463
          1.5545-2.0993 1.6557-7.6610 -10.89 7.2704 1.5813 1.8156 1.7166 -.4128
 16       -.0285  .2356  .7536 -.5630  .1775 -.3428  .6682-1.2603  .0715-1.9422
         -1.4499-1.4528-2.5209-1.4804-7.8211 10.95  7.0640-1.5133 1.4649-1.1589
 17        .0146 -.1386  .2071 -.9285-1.2376  .6357  .0268 -.0683 -.1566 -.6838
         -1.7305  .7931-1.7231 2.1817-1.8111 7.5334 10.35 -7.1214-2.5970  .0214
 18        .0298 -.1849 -.4155 -.0370 -.8306  .7518 -.3698  .9597 -.0386 1.1349
           .3312 1.3569  .9980 1.5198 2.2528 1.6537 7.5166 -10.44-7.1392 2.5484
 19        .0098  .0365 -.3901  .7224  .4341  .0629 -.1625  .6370  .0955 1.2808
          1.5295  .3162 1.8487 -.4458 2.2631-1.8361 2.8620-7.5414-8.1897 3.8976
 20       -.0063  .1252 -.0540  .5079  .6838 -.2814  .1147 -.0500  .0831  .3232
           .8484 -.3672  .7784 -.8595  .5804-1.5108 -.0132-2.8304-4.1103 3.2771
```

M_2

COLUMN VECTORS

ϵ_-/i	ϵ_+/i	$170\,\zeta_-$	$170\,\zeta_+$	δ_-	$B_o\epsilon_- + .170\,C_1\,\zeta_- + \delta_-$	$B_o\epsilon_+ + 170\,D_1\,\zeta_+$
-4080	4734	233	-128	-24930	-10430	-16796
8852	6526	1700	1700	- 1055	-28963	-17184
73	-914	-1296	961	4	2344	6222
3156	3293	33	72	- 334	-11212	-11171
119	-668	-531	379		- 1267	3519
1863	2001	20	43	- 204	- 6622	- 6790
137	-524	-342	242		- 1025	2584
1344	1443	14	30	- 143	- 4773	- 4901
104	-431	-253	179		- 767	2062
1054	1126	10	24	- 111	- 3744	- 3822
93	-372	-201	142		- 646	1740
858	913	9	19	- 89	- 3045	- 3101
97	-327	-167	119		- 605	1511
733	781	7	16	- 81	- 2608	- 2653
104	-304	-143	100		- 590	1370
622	687	6	14	- 65	- 2208	- 2333
129	-268	-125	88		- 648	1208
572	611	5	13	- 60	- 2032	- 2074
77	-256	-112	78		- 447	1134
550	529	6	10	- 46	- 1940	- 1799

```
                               GAMMA-   K1
 1       10000    629      1-    71            26       -    14             8
              -     5             4       -     3             2       -     2
 2       11076- 10000-  7848     22   1568-     7-   674      5    375-     2
       -   239      2    165       -   122            93       -    74
 3        1996- 15625- 10000-  9422          2244       -  1046           611
              -   492           287       -   244           167       -   134
 4        3646-    64  14126  10000  10093       -  2615          1282
       -   774           523       -   380           287       -   222
 5        1022-  6237         13457  10000  10468       -  2856          1452
              -   898           611       -   449           344       -   277
 6        2052-    36   5609       - 13079- 10000- 10692          3019
       -  1563           990       -   692           489       -   394
 7         688-  4040          5229       - 12837- 10000- 10874          3138
              -  1662          1060       -   745           559       -   430
 8        1538-    33   3662       -  5000         12692  10000  10986
       -  3227          1739       -  1120           803       -   602
 9         554-  2980          3419       -  4833         12550  10000  11079
              -  3305          1796       -  1165           840       -   630
10        1225-    21   2751       -  3267          4707       - 12465- 10000
       - 11154          3374       -  1848          1204       -   881
11         430-  2364          2579       -  3128          4608       - 12393
       - 10000- 11223          3415       -  1886          1242       -   907
12        1024-    26   2206       -  2469          3047       -  4544
         12345  10000  11269       -  3441          1917       -  1287
13         372-  1977          2092       -  2378          2980       -  4499
                12292  10000  11289       -  3496          1948       -  1289
14         870          1863       -  1988          2298       -  2919
          4441      - 12236- 10000- 11335          3509       -  1988
15         334-  1705          1772       -  1939          2240       -  2875
                 4380       - 12203- 10000- 11368          3544       -  2006
16         716-    36   1540       -  1683          1862       -  2185
          2830      -  4370         12178  10000  11390       -  3582
17         267-  1490          1528       -  1642          1833       -  2139
                 2788       -  4316         12147  10000  11421       -  3591
18         690          1420       -  1461          1583       -  1786
          2110      -  2760          4302       - 12135- 10000- 11445
19         258-  1289          1332       -  1418          1547       -  1762
                 2106       -  2750          4297       - 12119- 10000- 11474
20         590          1270       -  1316          1361       -  1497
          1724      -  2042          2722       -  4265         12114  10000
```

```
                             GAMMA+   K1
  1      10000-   240      4     36-     1-    13             7       -     4
                    3       -     2             1       -     1             1
  2    -   457- 10000   7085     96-  1415-    35    607     17-   337-    11
           216      7-   148-     4    109      4-    85-     2     65      2
  3       1218  14907- 10000   9443      5-  2261      5   1052       -   614
                  405       -   290           214       -   165           133
  4         14-   314- 14076  10000- 10057-    14   2605      7-  1278
           771      -    521           378      -    286           221
  5        611   5922-    10- 13429  10000- 10478          2856       -  1452
                  898       -   611           449       -   344           227
  6         36-   191-  5602     24  13063- 10000  10679       -  3016
          1561      -    989           691      -    513           393
  7        415   3811-    14-  5215         12837- 10000  10874       -  3138
                 1662       -  1060           745       -   559           430
  8         50-   134-  3662     17   5000       - 12692  10000- 10986
          3227      -   1739          1120      -    803           602
  9        305   2824       -  3397          4828       - 12538  10000- 11069
                 3302       -  1794          1164       -   840           630
 10         43-   108-  2753          3269      -   4710         12473- 10000
         11161      -   3376          1849      -   1204           882
 11        239   2249       -  2560          3134       -  4617         12416
       - 10000  11244       -  3421          1890       -  1244           909
 12         52-    79-  2205          2467      -   3045          4541
       - 12336  10000- 11286          3434      -   1916          1286
 13        201   1862       -  2092          2378       -  2980          4499
              - 12292  10000- 11289          3496       -  1948          1289
 14         31-    62-  1863          1988      -   2298          2919
       -  4441         12236- 10000  11335      -   3509          1988
 15        167   1605       -  1739          1940       -  2241          2876
              -  4381         12207- 10000  11371       -  3545          2007
 16         72-    72-  1613          1685      -   1864          2186
       -  2832          4373       - 12186  10000- 11398          3584
 17        229   1412       -  1527          1641       -  1832          2137
              -  2786          4313       - 12137  10000- 11412          3588
 18         41-    41-  1423          1463      -   1585          1789
       -  2114          2764       -  4309         12154- 10000  11463
 19        214   1245       -  1330          1416       -  1545          1760
              -  2103          2747       -  4292         12103- 10000  11459
 20         45-    45-  1273          1318      -   1364          1500
       -  1727          1864       -  2727          4273       - 12136  10000
```

```
                               GAMMA -   K1
  1         .9966  .0293  .0243  .0017  .0130 -.0011  .0087 -.0019  .0066 -.0023
            .0053 -.0025  .0047 -.0030  .0042 -.0034  .0039 -.0045  .0041 -.0096
  2        -.1438 1.3840 1.1470  .0400  .5226 -.0538  .3452 -.0805  .2599 -.0930
            .2091 -.0982  .1837 -.1182  .1623 -.1348  .1550 -.1777  .1626 -.3758
  3        -.0964 2.4177 1.2255 -.2055  .4398 -.0789  .3011 -.0815  .2298 -.0875
            .1863 -.0901  .1647 -.1076  .1458 -.1223  .1396 -.1607  .1466 -.3398
  4         .0180 -.1368  .3016 -.3864 -.6059  .0489 -.2508  .0554 -.1691  .0588
           -.1302  .0601 -.1118  .0709 -.0971  .0798 -.0922  .1045 -.0968  .2210
  5         .0975-2.2081 -.8825 -.8078 -.9072  .3919 -.3435  .1451 -.2651  .1201
           -.2162  .1135 -.1920  .1303 -.1704  .1454 -.1639  .1895 -.1734  .4003
  6         .0095 -.3083 -.2074 -.0879 -.4965  .3539  .4858 -.0502  .1661 -.0462
            .1030 -.0439  .0790 -.0486  .0648 -.0530  .0587 -.0676  .0587 -.1382
  7        -.0970 2.1702  .8983  .4971  .5103  .5897  .7722 -.4673  .2785 -.1850
            .2284 -.1436  .2036 -.1507  .1811 -.1620  .1746 -.2071  .1849 -.4326
  8        -.0256  .6130  .2940  .1325  .2592  .0704  .5487 -.3645 -.4432  .0391
           -.1290  .0336 -.0730  .0337 -.0507  .0350 -.0405  .0423 -.0370  .0827
  9         .0982-2.1745 -.9126 -.4464 -.5247 -.2721 -.3711 -.5089 -.7032  .5083
           -.2386  .2091 -.2135  .1814 -.1908  .1828 -.1839  .2249 -.1961  .4629
 10         .0396 -.8996 -.4011 -.1803 -.2751 -.0844 -.2813 -.0509 -.5756  .3780
            .4155 -.0255  .1055 -.0217  .0492 -.0190  .0291 -.0208  .0180 -.0323
 11        -.0990 2.1974  .9297  .4307  .5375  .2132  .3822  .1879  .2997  .4649
            .6594 -.5327  .2229 -.2486  .1992 -.2133  .1939 -.2483  .2065 -.4952
 12        -.0521 1.1869  .5171  .2299  .3260  .1002  .2733  .0543  .2958  .0315
            .5915 -.3920 -.3962  .0111 -.0844  .0058 -.0280 0.0000 -.0044 -.0155
 13         .1038-2.2661 -.9635 -.4342 -.5593 -.1941 -.3993 -.1274 -.3142 -.1430
           -.2606 -.4393 -.6480  .5741 -.2116  .2837 -.2065  .2856 -.2214  .5405
 14         .0706-1.5759 -.6815 -.2990 -.4124 -.1247 -.3195 -.0621 -.2893 -.0312
           -.3168 -.0151 -.6196  .4128  .3704  .0096  .0550  .0237 -.0084  .0749
 15        -.1063 2.3722 1.0108  .4483  .5892  .1898  .4219  .1054  .3337  .0799
            .2774  .1148  .2515  .3949  .6409 -.6143  .2233 -.3638  .2402 -.6081
 16        -.0902 2.1110  .9077  .3969  .5418  .1605  .4036  .0759  .3417  .0322
            .3176  .0103  .3555 -.0120  .6576 -.4419 -.3258 -.0550  .0002 -.1537
 17         .1236-2.5888-1.1069 -.4838 -.6459 -.1974 -.4644 -.0971 -.3673 -.0557
           -.3084 -.0495 -.2808 -.0624 -.2550 -.3458 -.6644  .7095 -.2726  .7393
 18         .1457-3.1663-1.3596 -.5899 -.8027 -.2339 -.5871 -.1048 -.4785 -.0405
           -.4206 -.0043 -.4079  .0332 -.4424  .0629 -.7555  .5253  .2071  .3056
 19        -.1444 3.0523 1.3068  .5671  .7647  .2268  .5526  .1024  .4401  .0424
            .3691  .0211  .3409 -.0200  .3084 -.0021  .3065  .2183  .7487-1.1942
 20        -.3519 7.4865 3.2115 1.3878 1.8881  .5437 1.3708  .2335 1.1013  .0752
            .9380 -.0098  .8685 -.1190  .8270 -.1969  .8803 -.3417 1.2645-1.3434
```

```
                                GAMMA  +   K1
  1      1.0437 -.0596  .0554 -.0029  .0285  .0030  .0191  .0046  .0144  .0053
          .0115  .0056  .0103  .0065  .0091  .0075  .0088  .0101  .0090  .0212
  2      -.7664 1.0219 -.9574  .0234 -.4294 -.0491 -.2819 -.0699 -.2110 -.0795
         -.1671 -.0820 -.1516 -.0960 -.1326 -.1104 -.1286 -.1474 -.1330 -.3104
  3      1.6861-1.9191 1.0014  .2219  .3312  .0723  .2274  .0682  .1735  .0710
          .1380  .0704  .1277  .0823  .1114  .0938  .1088  .1252  .1125  .2631
  4       .1317 -.0847 -.3265 -.3792  .5914  .0500  .2409  .0549  .1617  .0578
          .1257  .0614  .1032  .0664  .0922  .0767  .0862  .1000  .0898  .2098
  5     -1.7626 1.7300 -.6662  .7848 -.8004 -.3869 -.2709 -.1332 -.2095 -.1048
         -.1697 -.0965 -.1534 -.1053 -.1361 -.1184 -.1318 -.1545 -.1377 -.3242
  6       .2326 -.2356  .1761 -.0833  .4813  .3539 -.4967 -.0513 -.1744 -.0481
         -.1108 -.0478 -.0831 -.0496 -.0704 -.0556 -.0640 -.0722 -.0649 -.1491
  7      1.7684-1.7121  .6894 -.4787  .4079 -.5949  .7025  .4546  .2255  .1695
          .1840  .1266  .1669  .1253  .1488  .1342  .1456  .1741  .1521  .3602
  8      -.4978  .4820 -.2337  .1283 -.2310  .0721 -.5292 -.3606  .4575  .0436
          .1422  .0402  .0816  .0387  .0600  .0420  .0488  .0512  .0464  .1026
  9     -1.7825 1.7151 -.7044  .4269 -.4214  .2774 -.3012  .5211 -.6495 -.4931
         -.1939 -.1924 -.1764 -.1557 -.1584 -.1552 -.1545 -.1914 -.1631 -.3901
 10       .7367 -.7096  .3147 -.1732  .2333 -.0867  .2527 -.0560  .5533  .3716
         -.4353 -.0348 -.1189 -.0301 -.0627 -.0297 -.0412 -.0341 -.0316 -.0619
 11      1.8066-1.7314  .7181 -.4109  .4327 -.2185  .3113 -.2003  .2455 -.4810
          .6154  .5165  .1849  .2222  .1664  .1855  .1641  .2143  .1731  .4214
 12      -.9793  .9385 -.4045  .2204 -.2704  .1037 -.2357  .0615 -.2669  .0409
         -.5686 -.3814  .4145  .0229  .1019  .0199  .0437  .0177  .0220  .0232
 13     -1.8719 1.7899 -.7473  .4148 -.4531  .1998 -.3274  .1402 -.2593  .1592
         -.2149  .4560 -.6098 -.5481 -.1781 -.2555 -.1761 -.2511 -.1874 -.4654
 14      1.3036-1.2465  .5312 -.2873  .3400 -.1292  .2702 -.0714  .2515 -.0432
          .2856 -.0291  .5955  .3973 -.3936 -.0090 -.0757  .0004 -.0147  .0238
 15      1.9642-1.8751  .7853 -.4265  .4771 -.1955  .3462 -.1186  .2759 -.0965
          .2293 -.1320  .2108 -.4231  .6060  .5848  .1912  .3276  .2043  .5291
 16     -1.7530 1.6742 -.7091  .3794 -.4431  .1667 -.3371  .0885 -.2909  .0482
         -.2755  .0289 -.3227  .0095 -.6269 -.4169  .3534 -.0236  .0308 -.0850
 17     -2.1273 2.0311 -.8536  .4615 -.5220  .2036 -.3804  .1117 -.3032  .0738
         -.2549  .0678 -.2350  .0943 -.2158  .3796 -.6288 -.6696 -.2326 -.6513
 18      2.6226-2.5020 1.0568 -.5648  .6543 -.2430  .4868 -.1237  .4020 -.0643
          .3568 -.0316  .3578  .0001  .3958  .0247  .7128  .4779 -.2540  .2014
 19      2.5068-2.3902 1.0074 -.5375  .6173 -.2326  .4526 -.1183  .3636 -.0625
          .3052 -.0397  .2836 -.0203  .2615 -.0384  .2637 -.2682  .7011 1.0894
 20     -6.1877 5.9009-2.4921 1.3240-1.5352  .5625-1.1317  .2761 -.9186  .1289
         -.7855  .0490 -.7438 -.0352 -.7154 -.1042 -.7786 -.2271-1.1504-1.0940
```

K_1

COLUMN VECTORS

ϵ_-/i	ϵ_+/i	δ_-
- 823	-1871	-38495
18025	11594	- 1167
6103	4154	- 1189
3675	2503	- 716
2609	1773	- 518
2364	1393	- 387
1944	1129	- 315
1677	932	- 280
1468	1290	- 251
1339	1098	- 203
1180	1000	- 181

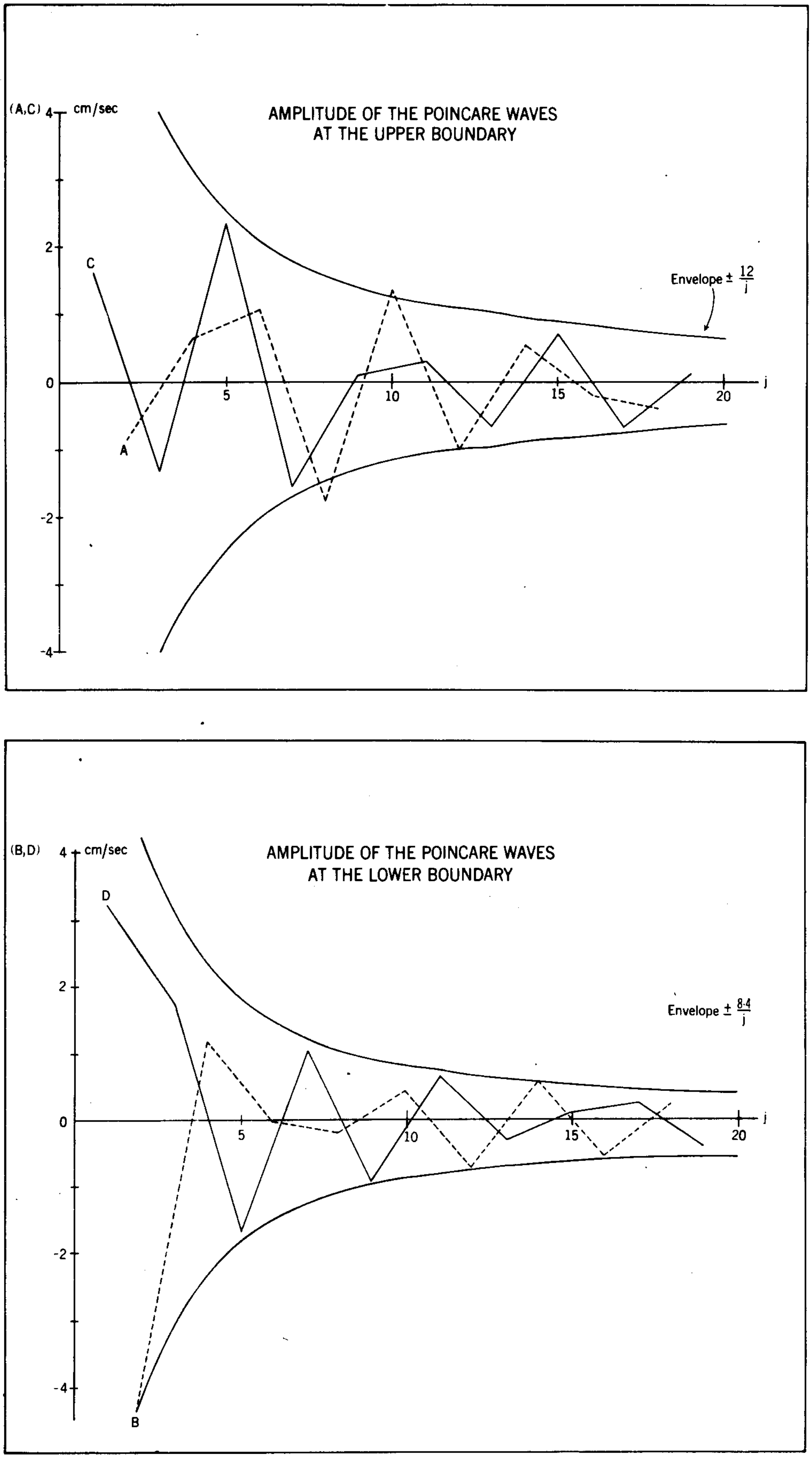

(A,C)
cm/sec
4
2
0
-2
-4
AMPLITUDE OF THE POINCARE WAVES
AT THE UPPER BOUNDARY
C
A
Envelope ± $\frac{12}{j}$
5
10
15
20
j
(B,D)
cm/sec
4
2
0
-2
-4
AMPLITUDE OF THE POINCARE WAVES
AT THE LOWER BOUNDARY
D
B
Envelope ± $\frac{8.4}{j}$
5
10
15
20
j

Fig. 10

The Solution for M_2

Following the procedure outlined on p. 76 ff., we obtain the values of the arbitrary constants (A_n, C_m), (B_n, D_m) in 2 for M_2; their variation with j is illustrated in Fig. 10. It can be noticed that these quantities decrease at least as $(-)^j/j$ if not more rapidly. (The constant chosen for the envelope is arbitrary).

No such convergence is to be found for K_1. If we inspect Γ_+^{-1} and Γ_-^{-1} for K_1 on p. 115 and 116 we note that the second element of each row increases with j (alternate rows). Since this element is multiplied by the second element of the column vector ϵ on p. 117 (this element being by far the most important element of the column vector on the right) in order to obtain the further constants, once B_o is evaluated, it will have as a consequence that the Poincaré waves become larger and larger as j increases so that v will oscillate violently at the inner boundaries. This means that a convergent solution for K_1 does not exist. The reasons for this disappointing conclusion will be investigated once we have carried through our solution for M_2.

Going back to M_2, we notice that if the Poincaré waves vary as $(-)^j/j$ in 2, this will cause v to diverge logarithmically at $(\pm a_2, \pm l_2)$. In fact, using the 20 known constants, the plot of v at $\pm l_2$ (not illustrated) shows that it has large oscillations in the vicinity of $\pm a_2$ and we may surmise that these oscillations are the indication of such a type of divergence. Z on the other hand is finite and smooth everywhere.

This situation was to be expected considering the sharply discontinuous character of the inner corners. This however violates the condition we had set on p. 57 for the representation of discontinuous functions. $V(\pm l_2)$ can only represent a function which has a finite discontinuity at $\pm a$. It makes it then impossible, strictly speaking, to find solutions in 1 and 3 from those in 2: unless we find a way of removing this infinite discontinuity.

If we go back to equations (43) and (44) which describe the boundary condition on v at $\pm l_2$, we notice that the Kelvin waves in 1 and 3 are determined independently of the Poincaré waves in 2; they depend exclusively on $A_o^{\textcircled{2}}$ and $B_o^{\textcircled{2}}$. This provides a clue to the solution.

We are given the following facts:

1) The Poincaré waves only are responsible for the divergence at the corners.

2) The Kelvin waves are determined everywhere once they are known in 2, divergence or not.

Table VIII

SEA	1		2		3		
J	(A, C)	(B, D)	(A, C)	(B, D)	(A, C)	(B, D)	
0	-1.26	.27	-1.52	3.46	1.74	.00	
1	.29		1.61	3.18	- .49	.21	
2	.40		- .89	-4.21	.00	- .77	
3	- .29		-1.33	1.76	.06	.39	
4	.09		.65	1.15	.00	.39	
5	- .12		2.40	-1.67	- .02	- .19	
6	- .07		1.07	- .01	.00	- .29	
7	.02		-1.56	1.05	- .01	.11	
8	- .03		-1.80	- .19	.00	.15	
9	.07		.10	- .90	.01	- .05	
10	.05		1.35	.45	.00	- .02	
11	- .01		.33	.68	.00	- .01	
12	.03		- .98	- .69	.00	- .07	
13	- .04		- .61	- .28	.00	.05	
14	- .03		.58	.61	.00	.12	
15	.00		.71	.14	.00	- .06	
16	- .03		- .18	- .50	.00	- .11	
17	.04		- .65	.33	.00	.05	
18	.01		- .25	.30	.00	.07	
19	.01		.14	- .35	.00	.02	CM/SEC

We know as well that the singularity is to be found at $\pm\ l_2$ and not within 2 due to presence of the exponential damping factors.

We use this damping feature in the following way: we equate hv at l_1 and $-\ l_3$ to a hv in 2 made up of the superposition of the Kelvin waves at the inner boundaries $\pm\ l_2$ and the contribution of the Poincaré waves at some point within 2.

In this way we moderate the effect of the corners in 1 and 3 and we may obtain solutions for the two seas. This makes physical sense since infinite velocities do not exist and equating infinite transports only leads to indeterminate solutions.

The only element of arbitrariness is the distance we choose within 2 to pick the Poincaré waves; this will have as a consequence that the solution for v and Z in the immediate vicinity of $+\ l_1$ and $-\ l_3$ will vary slightly with our choice. But this indeterminacy does not travel very far in 1 and 3 where the solution rapidly becomes unique.

The point we choose where to evaluate the contribution of the Poincaré waves to hv is $y_2 = 0$; this is an obvious choice since it lies halfway between $\pm\ l_2$ and sea 2 is very short compared to 1 and 3.

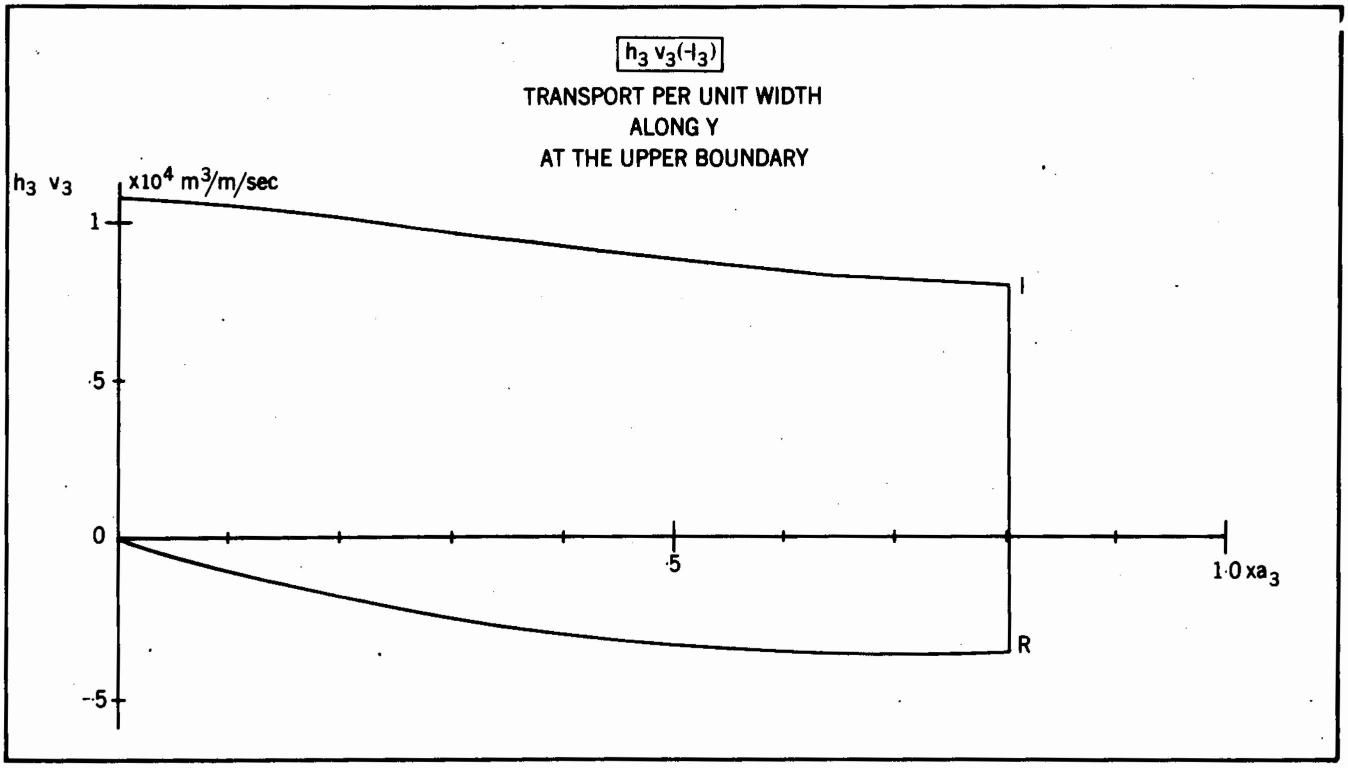
In this way we obtain a complete set of values for $(A_n^{①}, C_m^{①})$, $(B_n^{③}, D_m^{③})$ which are tabulated in Table 8 along with the other constants.

The transports per unit width and the elevations at the inner boundaries are shown in Fig. 11 and 12. Some discontinuity will be noticed in the elevations since we have contradicted the solution obtained in 2 where we had assumed the Poincaré waves in 2 to be balanced at $\pm\ l_2$. These discrepancies are slight and could be removed completely if the computations were repeated taking into account the modifications introduced into the boundary condition on v. There are no discrepancies in the transports since $(A_n^{①}, C_m^{①})$ and $(B_n^{③}, D_m^{③})$ have been chosen to satisfy the modified boundary condition exactly.

Charts showing Z, hv and the elements of the current ellipses for M_2 are given in Fig. 13 and 14.

Appendix 2 Part 3 gives a comparison between the transport bhv ($Z_0 = 70$cm) for the 6 channel model and that for the 3 basin model.

The solution given in Table 8 has been multiplied by the factor 7.5 in order that the value of Z_0 be approximately 70 cm across the mouth.


$h_3 v_3(+_3)$
TRANSPORT PER UNIT WIDTH
ALONG Y
AT THE UPPER BOUNDARY
$h_3 v_3$
$x10^4$ m^3/m/sec
1
·5
0
-·5
·5
1·0 xa_3
I
R

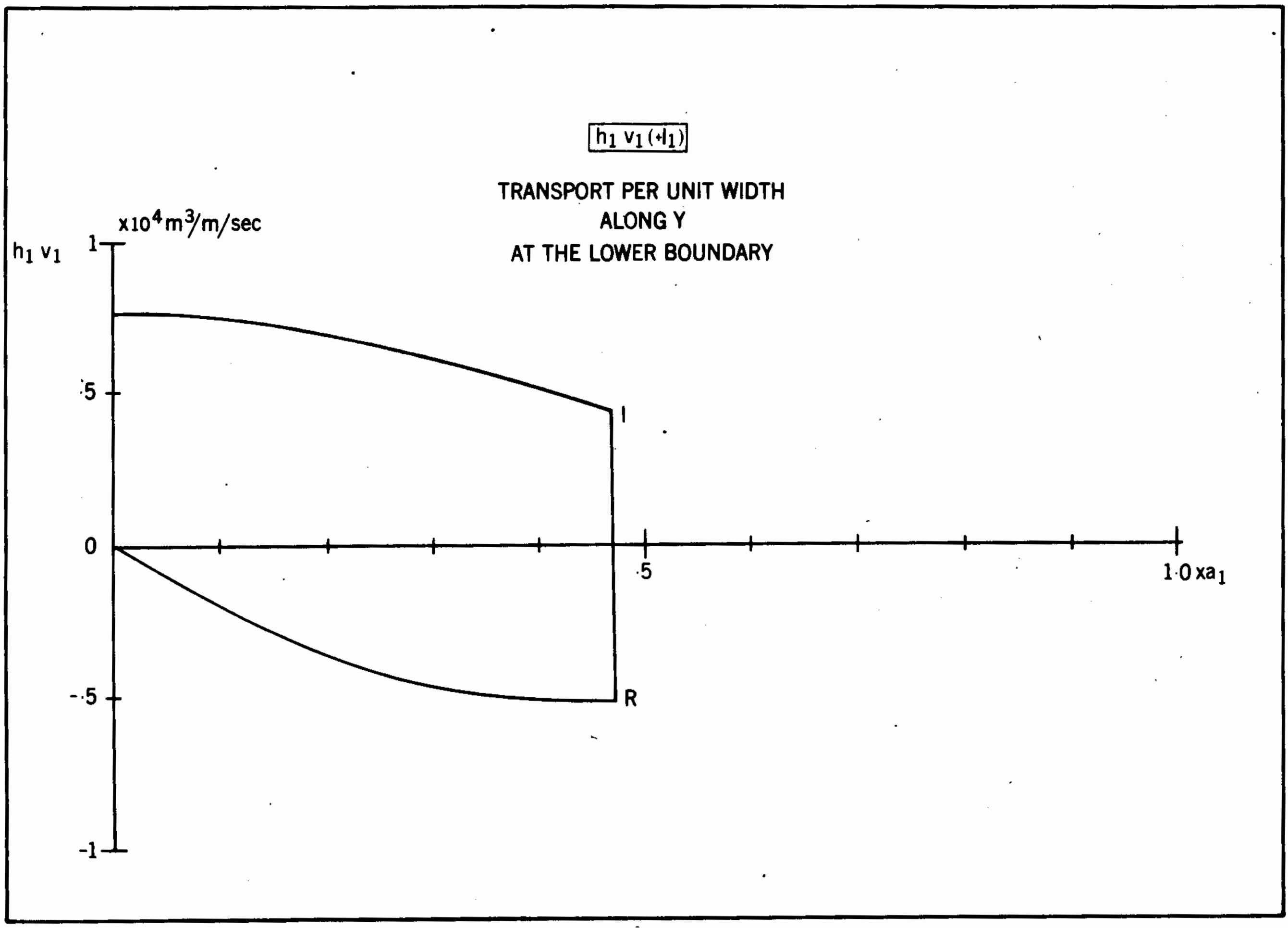

$h_1 v_1(+_1)$
TRANSPORT PER UNIT WIDTH
ALONG Y
AT THE LOWER BOUNDARY
$h_1 v_1$
$x10^4$ m^3/m/sec
1
·5
0
-·5
-1
·5
1·0 xa_1
I
R

Fig. 11

9.11

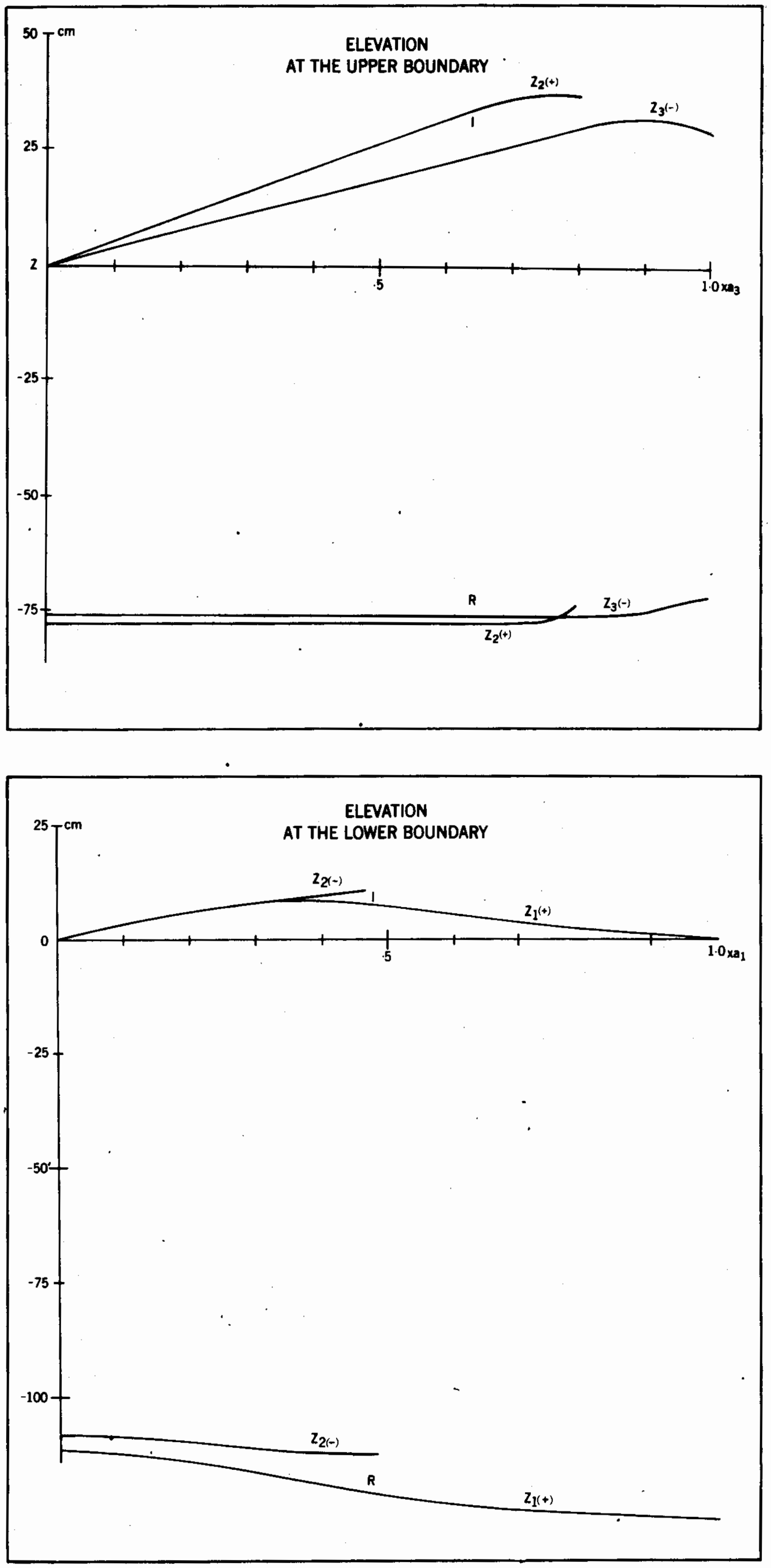

ELEVATION
AT THE UPPER BOUNDARY
50
cm
25
Z
-25
-50
-75
·5
1·0 xa₃
Z₂(+)
Z₃(-)
I
R
Z₃(-)
Z₂(+)
ELEVATION
AT THE LOWER BOUNDARY
25
cm
0
-25
-50
-75
-100
·5
1·0 xa₁
Z₂(-)
I
Z₁(+)
Z₂(-)
R
Z₁(+)

Fig. 12

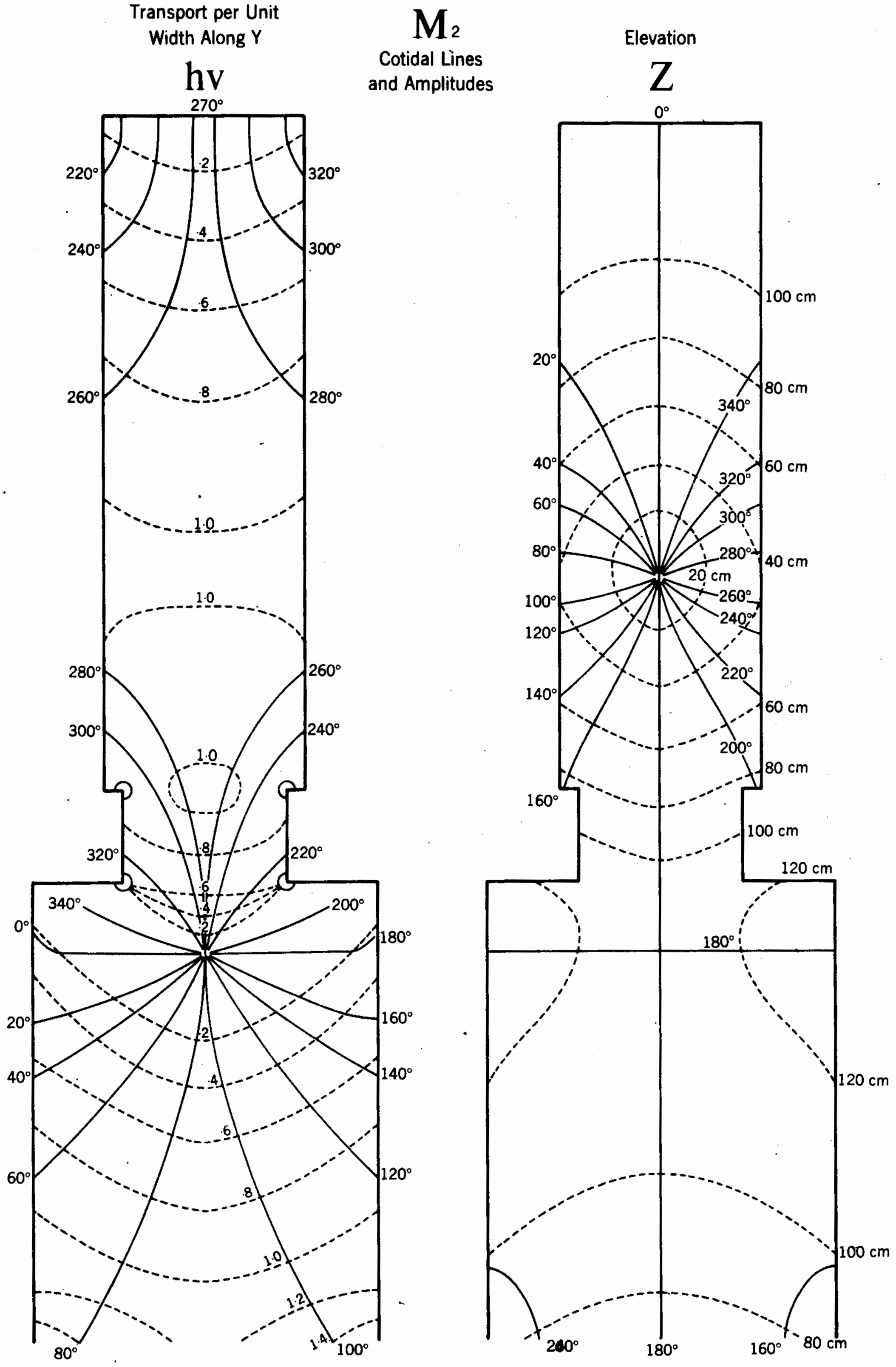

Transport per Unit Width Along Y
hv
M2
Cotidal Lines and Amplitudes
Elevation
Z
270°
220°
320°
240°
300°
260°
280°
280°
260°
300°
240°
320°
220°
340°
200°
0°
180°
20°
160°
40°
140°
60°
120°
80°
100°
·2
·4
·6
·8
1·0
1·0
1·0
·8
·6
·4
·2
·2
·4
·6
·8
1·0
1·2
1·4
0°
20°
340°
40°
320°
60°
300°
80°
280°
100°
260°
120°
240°
140°
220°
200°
160°
180°
200°
180°
160°
20 cm
40 cm
60 cm
80 cm
100 cm
60 cm
80 cm
100 cm
120 cm
120 cm
100 cm
80 cm

Fig. 13

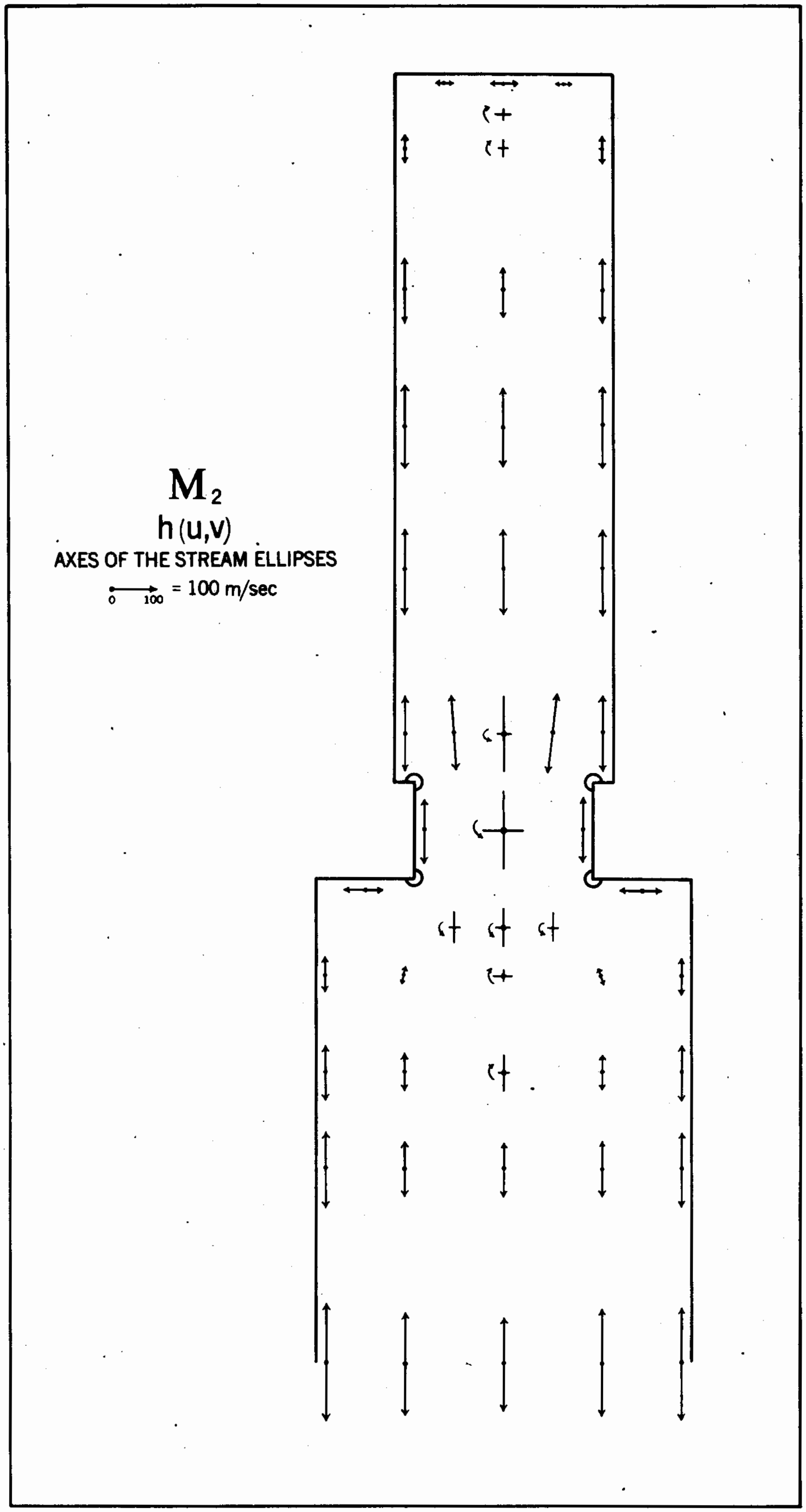


Fig. 14

Discussion of the Results for M_2

Once again theory supplies us with more information than that given by observations. Not only do we obtain a cotidal chart for the elevation but as well a cotidal chart for hv (the most important part of the transport) and a complete map of the tidal ellipses.

If we compare the cotidal chart for Z with the one based on observations (Fig. 2) we notice a remarkable qualitative agreement between the two charts. This agreement is most remarkable when we consider the crudeness of the model used to represent the sea. As a matter of fact the points of disagreement between the two charts indicate the effects of the features not taken into account in the model and to be found in the actual sea.

The 160° line in the model curving on the Greenland coast has its counterpart in the empirical chart. The 180° line is seen in the model to go straight up to the node while another 180° line goes across the Sea of Labrador just ahead of Davis Strait. Coastal observations allow the two same 180° lines to be drawn in the empirical chart. The 200° line in the theoretical chart has its counterpart in the empirical chart but there we see that nearer the coast it is followed by a 220° and a 240° line. These indicate the effect of the important shelf lying off the Labrador coast which had been neglected in the model.

In Baffin Bay, the theoretical chart puts the 90° and 270° lines straight across while in the empirical chart they are in the shape of an inverted V. More, in the empirical chart the cotidal lines are more crowded in the lower part of the Bay. These two features reflect the fact that in the model the depth in the Bay decreases discontinuously at its junction with Davis Strait while in actuality the change is gradual and causes some slowing down in the progress round the node in this area. The position of the 000° line underlines the asymmetry in the basin which has not been taken into account in the model.

When it comes to the amplitudes it can be noticed that the two 120 cm lines in the calculated chart have their counterpart in the observed chart. However there is in reality a considerable amplification of the tide on the Canadian side of the sea which the model cannot reproduce. The overall agreement in the amplitudes is quite good.

We cannot compliment ourselves on the position of the node since the length of sea 3 has been chosen in such a way that it would fall where expected. The last 300 km of sea 3 in the model are fictitious and aim at representing the effect of Smith Sound. It should be noted that in the actual sea the node is closer to Baffinland; the bathymetric chart shows that it lies approximately where the greatest depth across the section is to be found.

The Divergence at the Inner Boundaries for K_1

It does not seem easy at first sight to unravel from the complicated expressions we have been handling the reason why the solution for K_1 diverges. However a bit of analysis will reveal that it is essentially related to the fact that

$$\sigma < 2\omega$$

for K_1 due to the high latitude of the sea under study.

If we go back to equations (48) and look at the order of magnitude of the terms as $j \rightarrow \infty$ we notice first that the interaction terms already tabulated in Tables 5 and 6 decrease at least as $1/j^3$ and can therefore be neglected in comparison with terms decreasing as $1/j$ or $1/j^2$.

If we choose to look at the equation corresponding to k = 3 (the simplest to handle: the reasoning would have been identical had we chosen k = 1) we obtain expressions of the form

$$\alpha_2 \left[\sigma \frac{A_n^{\textcircled{2}}}{K_n} - 2\omega \sum \frac{P_{mn}}{m} \; C_m^{\textcircled{2}} \right] \simeq A_o^{\textcircled{3}} \left(\frac{h_3}{g}\right)^{\frac{1}{2}} u_n^{\textcircled{3}} \cos 2K_o^{\textcircled{3}} l_3 - \left(\frac{h_2}{g}\right)^{\frac{1}{2}} p_n^{\textcircled{2}} \, A_o^{\textcircled{2}}$$

$$i\alpha_2 \left[\sigma \frac{C_m^{\textcircled{2}}}{K_m} - 2\omega \sum \frac{Q_{nm}}{n} \; A_n^{\textcircled{2}} \right] \simeq B_o^{\textcircled{2}} \; q_m^{\textcircled{2}} \left(\frac{h_2}{g}\right)^{\frac{1}{2}} - iA_o^{\textcircled{3}} \left(\frac{h_3}{g}\right)^{\frac{1}{2}} V_m^{\textcircled{3}} \sin 2K_o^{\textcircled{3}} l_3$$

as $j \longrightarrow \infty$

if we care to remember the definitions of $G^{\textcircled{3}}$ and $A_o^{\textcircled{3}}$.

These equations simply state that the difference between the Kelvin waves in 2 and 3 should be balanced by the Poincaré waves in 2 when $j \gg 1$. This makes physical sense since the Kelvin waves are the main mode of motion and that the presence of a boundary will create Poincaré waves whose only purpose is to balance the Kelvin waves and thus satisfy the boundary condition. Also we have already noticed in the solution of M_2 that the Poincaré waves in 1 and 3 diminish much more rapidly than the Poincaré waves in 2.

We notice next that

$$p_n, \; u_n, \; q_m, \; v_m \text{ decrease as } 1/j^2$$

while

$$1/K_j \sim 1/j$$

and $|P_{mn}| \;/m, \quad |Q_{nm}| \;/n \sim 2/\pi j$ for $m = n \pm 1, \; n = m \pm 1$
$j >> 1$

(59) is then equivalent to the relations

$$\ldots A_{m-1} + C_{m-2} + A_{m+1} \ldots = \text{const}/m$$

$$\ldots C_{n-1} + A_{n-} + C_{n+1} \ldots = \text{const}/n \qquad (60)$$

The further terms on the left and the right indicated by dots are affected by the factors 1/3, 1/5, 1/7 etc.

If we inspect the 3 main elements of each row of the matrices $\Gamma_{\pm}$ for K_1 on p. 115, 116 we note that the factor affecting the constant in $j - 1$ is consistently larger than the one affecting the constant in $j + 1$ but that as $j \rightarrow \infty$ both tend to become equal and in constant ratio to the diagonal element. This ratio gives the value of $(2/\pi) \; / \; (\sigma/2\omega)$. The column vector (renormalized) on p. 119 decreases as $1/j$.

Now suppose we solve a set of 6 equations of the form (60) by successive elimination from below, assuming j to be so large that the right hand side is approximately zero, so that all the unknowns are expressed in terms of the one of smallest order. We obtain two distinct sets of values for the unknowns depending on whether the ratio $r = (\sigma/2\omega) \; / \; (2/\pi)$ is smaller or larger than 1.

Taking A_j as the unknown of lowest order and trunkating C_{j+7} away (This is always so: the term just to the right of the last diagonal element is always removed when we give finite dimensions to the matrices), we obtain:

	$r << 1$	$r >> 1$
A =	A_j	A_j
C_{j+1} =	$r A_j$	A_j/r
A_{j+2} =	A_j	A_j/r^2
C_{j+3} =	0	$-A_j/r^3$
A_{j+4} =	$-A_j$	A_j/r^4
C_{j+5} =	$-rA_j$	$-A_j/r^5$
A_{j+6} =	$-A_j$	$-A_j/r^6$

It can be seen that for $r << 1$ the unknowns just oscillate around A_j while for $r >> 1$ the convergence gets stronger and stronger as we move into the higher orders.

The above calculations are very rough but indicate the crucial role played by r. The criterion for convergence seems to be

$$\sigma/2\omega > 2/\pi = .64$$

and a more stringent form would be

$$\sigma/2\omega > 1$$

There is no way of checking this analytically due to the great algebraic complication. But even if we take the criterion for convergence in its weakest form there is little chance of getting a convergent solution for any diurnal constituent. For OO_1 the fastest diurnal constituents (σ = 16.1°/hour),

$$\sigma/2\omega = .60 < .64$$

Unless we choose to bring the sea to a more southern latitude no convergent solution exists for the diurnal tides.

The above considerations give the mathematical reason for the divergence. There is no obvious physical interpretation for such a state of things.

The source of the difficulty can be traced to (27) expressing Z in a form which is such that $u(\pm a) = v$ is satisfied.

Had we written Z directly in terms of the basic vectors introducing a new set of arbitrary constants affecting these vectors we could have evaluated these constants easily from the boundary condition on the elevations.

On the other hand writing the solution for Z in such a form would result in an expression for u which would violate the above mentioned boundary condition. Since the whole formulation of our problem rests on the logical consequences deriving from the satisfaction of this boundary condition (eigenvalues, eigenfunctions, basic sets), we cannot throw all this away just for the sake of getting an answer.

This leaves us without any solution for K_1 but from the results obtained for M_2 , we may make some conjectures about it.

First the one dimensional calculations give a very good clue about the two dimensional solution once we have translated the trigonometric waves into Kelvin waves. For instance, for M_2 , the position of the node found in one dimension is the same as the one found in two dimensions and the profile of elevation along the central line of the sea is given closely by the one dimensional profile except in and around Davis Strait.

For K_1 we may assume that we would have obtained a node in the Strait, relatively small elevations in 1 and larger elevations in 3.

This however would not compare well with Fig. 3 where the node lies on Cumberland Peninsula. This brings to our attention the fact that the model we chose is completely symmetrical in x while it is obvious that the actual sea does not exhibit such complete symmetry; sea 1 and 3 would have been more representative if we had brought them more to the left of the central line of symmetry in 2.

This asymmetry does not seem to affect M_2 appreciably since nothing crucial happens to this constituent in 2. However for K_1 such an asymmetry should have a marked influence on the position of the node and the projection of

Cumberland Peninsula into the Strait must be responsible for the disturbance in the position of the point of amphidromy.

GENERAL CONCLUSIONS

We have investigated with the help of theory the tidal regime in the sea made up of Baffin Bay, Davis Strait and the Labrador Sea.

Our study has shown that the tides are induced mainly by those in the Atlantic; also the motion seems to be of a standing wave character which is interpreted as the perfect reflection of a travelling wave at the head of the sea.

The shape of the basin suggested immediately that linear motion should predominate. Accordingly one dimensional calculations were carried out which supported well the general variation in amplitude along the sea, once proper account was taken of the diffuse reflection which occurs at the head.

A by-product of this investigation was values for the mean currents across the sea. Such values have been confirmed by two dimensional calculations for M_2 and therefore give some indication on the horizontal motion prevailing in such a sea.

Two dimensional calculations over the same basin proved to be exceedingly difficult in spite of the drastic schematization of the width and depth and lead to satisfactory results only when $\sigma < 2\omega$, otherwise divergence occurs at the inner boundaries which makes absurd any attempt at satisfying inner boundary conditions. When convergence occured however the results were most rewarding and redeemed the effort spent.

For the semidiurnal tide M_2 , we conclude that:

1) There is a maximum amplitude in elevation just ahead of Davis Strait,

2) A node is found at latitude 70°N nearer to Baffinland than to Greenland,

3) High water is simultaneous along the center of the sea till one reaches the node where there is a change of phase of 180°,

4) The tide progresses very slowly round the node in the southern section of Baffin Bay,

5) The very large tides experienced in the vicinity of Hudson Strait are most likely prevalent only in the immediate vicinity of the coast,

6) The currents are alternating over most of the sea; marked elliptic currents are to be found on the Labrador Shelf, in the vicinity of Davis Strait and at the head of Baffin Bay,

7) The maximum currents over the body of the sea are found in Davis Strait and are of the order of 15 cm/sec,

8) The weakest currents are to be found at the southern opening of Davis Strait,

9) The current at the mouth of the Sea of Labrador is of the order of 10 cm/sec,

10) The gradients in the elevation and the currents across a section are due mainly to the gradients inherent to the superposition of two standing Kelvin waves of an odd and even character since the motion over the body of the sea is predominantly Kelvin.

The areas where this does not hold are

a) The Labrador Shelf where large currents and elevations are induced by the topography,

b) The entrance (64°N) and exit (67°N) of Davis Strait.

The conclusions for the diurnal tide K_1 are not so categorical due to the difficulties encountered during the calculations. We state

1) The diurnal elevations are small over Davis Strait and the Labrador Sea; they become important in the northern sections of Baffin Bay,

2) Diurnal streams are important in Davis Strait and in the northern section of the Labrador Sea; they most likely predominate over the semidiurnal streams in the latter area while they are of the same order of magnitude as the semiurnal streams in the Strait itself.

The same conclusions hold for the other diurnal and semi diurnal tides; the numerical values mentioned above should be reduced by factors representing the ratio of the amplitude of these tides to that of K_1 or M_2.

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Table I

Tidal Stations. Name, geographical location, duration of the interval of observations, authority having performed the analysis.
(C. H. S. - Canadian Hydrographic Service)

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APPENDIX 1

SOLUTIONS OF EQUATIONS (3) AND (4)

The differential equations in Z resulting from equations (3) and (4) is

$$\frac{1}{b}\frac{d}{dy}\left[b\,h\,\frac{dZ}{dy}\right]+\frac{\sigma^2}{g}\,Z=0 \qquad (1-1)$$

and v is related to Z through the relation

$$v=-\frac{g}{\sigma}\frac{dZ}{dy} \qquad (1-2)$$

1. - Constant Width and Depth

The laws of width and depth are

$$b=b_o \qquad h=h_o$$

(1-1) takes the form

$$\frac{d^2Z}{dy^2}+K^2Z=0 \qquad (1-3)$$

where

$$K\equiv\frac{\sigma}{(gh)^{\frac{1}{2}}}$$

This equation has for general solution

$$Z=A\cos Ky+B\sin Ky \qquad (1-4)$$

implying

$$v=\left(\frac{g}{h}\right)^{\frac{1}{2}}\left[A\sin Ky-B\cos Ky\right] \qquad (1-5)$$

A and B are arbitrary constants of integration; this will also be the case for the further solutions that will follow.

2. - Exponential Variation in Width and Depth

The y dependence of b and h is given by

$$b = b_o \, . \, e^{\beta y} \qquad\qquad h = h_o \, . \, e^{\alpha y}$$

(1-1) takes the form

$$\frac{d^2Z}{dX^2} - \frac{1}{X}\,\frac{\beta}{\alpha}\,\frac{dZ}{dX} + \frac{\sigma^2}{gh_o\alpha^2} \cdot \frac{Z}{X} = 0$$

through the substitution

$$X = e^{-\alpha y}$$

This equation is related to Bessel differential equation and it has for solution (Kamke (14))

$$Z = \left(\frac{h_o}{h}\right)^{\frac{1}{2}\left(1+\frac{\beta}{\alpha}\right)} \cdot \left[A\,J_{\left[1+\frac{\beta}{\alpha}\right]}(x) + B\,J_{-\left[1+\frac{\beta}{\alpha}\right]}(x) \right] \qquad (1\text{–}6)$$

implying

$$v = \left(\frac{h_o}{h}\right)^{\left(1+\frac{\beta}{2\alpha}\right)} \cdot \left(\frac{g}{h_o}\right)^{\frac{1}{2}} \cdot \left[AJ_{\beta/_\alpha}(x) - BJ_{-\beta/_\alpha}(x) \right] \qquad (1\text{–}7)$$

where the J's are Bessel functions of the first kind and of order $\frac{\beta}{\alpha}$ or $1+\frac{\beta}{\alpha}$ and where $x \equiv \frac{2\sigma}{\alpha(gh)^{\frac{1}{2}}}$ denotes the argument.

As long as $\frac{\beta}{\alpha}$ is not and integer, these series can be evaluated with the help of the series expansion

$$J_r(x) = \left(\frac{x}{2}\right)^r \sum_{j=o}^{\infty} \frac{(-)^j}{j!\,\Gamma(r+j+1)} \cdot \left(\frac{x}{2}\right)^{2j}$$

where Γ is the Gamma function.

For the special case $\beta = 0$ and $\alpha \neq 0$, the above solutions are in terms of Bessel functions of order 1 and 0, of the first and second kind, with a plus sign affixed in front of B in the formula for v.

For the case $\beta \neq 0$, $\alpha = 0$, the equation for Z becomes a homogeneous differential equation with constant coefficients which has for solution

$$Z = e^{-\frac{\beta y}{2}} \left[A e^{\lambda y/2} + B e^{-\lambda y/2} \right] \qquad (1-8)$$

implying

$$v = \frac{g}{2\sigma} \cdot \left[(\beta - \lambda) A e^{\lambda y/2} - (\beta + \lambda) B e^{-\lambda y/2} \right] e^{-\beta y/2} \qquad (1-9)$$

where

$$\lambda = \left(\frac{4\sigma^2}{g h_o} - \beta^2 \right)^{\frac{1}{2}}$$

For instance Channel 3 in the 6 Channel Model could be schematized by a channel of constant depth and exponential width.

APPENDIX 2

Part 1

SOLUTIONS FOR THE ONE DIMENSIONAL MODELS

1 Channel

$$Z = \frac{\cos K(y - l)}{\cos Kl} \qquad v = \left(\frac{g}{1120}\right)^{\frac{1}{2}} \frac{\sin K(y - l)}{\cos Kl}$$

where	M_2	K_1
$Kl =$	180.7°	93.7°

2 Channels

$$Z_1 = \left(\frac{310}{h_1}\right)^{.7763} \left(A_1 J_{1.5526}(x_1) + B_1 J_{-1.5526}(x_1)\right)$$

$$Z_2 = B_2 \cos K_2 (y - l_2)$$

$$v_1 = \left(\frac{310}{h_1}\right)^{1.2763} \left(\frac{g}{310}\right)^{1/2} \left(A_1 J_{.5526}(x_1) - B_1 J_{-.5526}(x_1)\right)$$

$$v_2 = B_2 \left(\frac{g}{310}\right)^{1/2} \sin K_2 (y - l_2)$$

where

	M2	K1	UNITS
A_1 =	.63	-9.29	CM
B_1 =	-4.35	-2.18	CM
B_2 =	- .89	-3.15	CM
X_1 =	.9456 → 2.7349	.4902 → 1.4177	
$K_2 l_2$ =	110.2°	59.7°	

3 Channels

$$Z_1 = \left(\frac{310}{h_1}\right)^{.6968} (A_1 J_{1.3936}(x_1) + B_1 J_{-1.3936}(x_1))$$

$$Z_2 = \left(\frac{310}{h_2}\right)^{-.1778} (A_2 J_{-.3555}(x_2) + B_2 J_{.3555}(x_2))$$

$$Z_3 = B_3 \cos K_3 (y - l_3)$$

$$v_1 = \left(\frac{310}{h_1}\right)^{1.1968} \left(\frac{g}{310}\right)^{\frac{1}{2}} (A_1 J_{.3936}(x_1) - B_1 J_{-.3936}(x_1))$$

$$v_2 = \left(\frac{310}{h_2}\right)^{.3223} \left(\frac{g}{310}\right)^{\frac{1}{2}} (A_2 J_{-1.3555}(x_2) - B_2 J_{1.3555}(x_2))$$

$$v_3 = \left(\frac{g}{800}\right)^{\frac{1}{2}} B_3 \sin K_3 (y - l_3)$$

where

	M2	K1	UNITS
A_1 =	.25	-6.98	CM
B_1 =	-3.46	-1.99	CM
A_2 =	-3.11	-6.21	CM
B_2 =	-2.90	1.01	CM
B_3 =	-1.26	-2.79	CM
X_1 =	.8281 → 2.3939	.4293 → 1.2409	
X_2 =	2.8735 → 2.5330	1.4896 → 1.3133	
$K_3 l_3$ =	110.2°	59.7°	

6 Channels

$$Z_1 = \left(\frac{1600}{h_1}\right)^{.6985} \left(A_1 J_{1.3969}(x_1) + B_1 J_{-1.3969}(x)\right)$$

$$Z_2 = \left(\frac{310}{h_2}\right)^{.6967} \left(A_2 J_{1.3934}(x_2) + B_2 J_{-1.3934}(x_2)\right)$$

$$Z_3 = \left(\frac{310}{h_3}\right)^{-.1778} \left(A_3 J_{-.3555}(x_3) + B_3 J_{.3555}(x_3)\right)$$

$$Z_4 = \left(\frac{400}{h_4}\right)^{.7950} \left(A_4 J_{1.5918}(x_4) + B_4 J_{-1.5918}(x_4)\right)$$

$$Z_5 = A_5 \cos K_5 y + B_5 \sin K_5 y$$

$$Z_6 = \left(\frac{250}{h_6}\right)^{.6994} \left(A_6 J_{1.3987}(x_6) + B_6 J_{-1.3987}(x_6)\right)$$

$$v_1 = \left(\frac{1600}{h_1}\right)^{1.1965} \left(\frac{g}{1600}\right)^{\frac{1}{2}} (A_1 J_{.3969}(x_1) - B_1 J_{-.3969}(x_1))$$

$$v_2 = \left(\frac{310}{h_2}\right)^{1.1967} \left(\frac{g}{310}\right)^{\frac{1}{2}} (A_2 J_{.3934}(x_2) - B_2 J_{-.3934}(x_2))$$

$$v_3 = \left(\frac{310}{h_3}\right)^{.3228} \left(\frac{g}{310}\right)^{\frac{1}{2}} (A_3 J_{-1.3555}(x_3) - B_3 J_{1.3555}(x_3))$$

$$v_4 = \left(\frac{400}{h}\right)^{1.2959} \left(\frac{g}{400}\right)^{\frac{1}{2}} (A_4 J_{.5918}(x) - B_4 J_{-.5918}(x_4))$$

$$v_5 = \left(\frac{g}{800}\right)^{\frac{1}{2}} (A_5 \sin K_5 y - B_5 \cos K_5 y)$$

$$v_6 = \left(\frac{250}{h_6}\right)^{1.1994} \left(\frac{g}{250}\right)^{\frac{1}{2}} (A_6 J_{.3987}(x_6) - B_6 J_{-.3987}(x_6))$$

Where	WITHOUT OUTFLOW		WITH OUTFLOW		
	M2	K1	M2	K1	UNITS
A_1 =	2.68	- 2.93	3.04	-2.11	CM
B_1 =	.06	- 2.45	.52	-2.17	CM
A_2 =	-3.03	- 10.90	-2.45	-8.07	CM
B_2 =	-2.40	- .63	-2.56	- .69	CM
A_3 =	-2.69	- 10.50	-3.45	-7.83	CM
B_3 =	-2.04	2.42	- .93	2.40	CM
A_4 =	2.87	16.80	2.32	11.80	CM
B_4 =	.81	2.92	- .29	1.84	CM
A_5 =	- .15	- 3.36	.42	-2.07	CM
B_5 =	- .82	- 2.94	- .83	-2.17	CM
A_6 =	1.30	3.29	2.03	.93	CM
B_6 =	1.03	1.41	.43	.63	CM
OUTFLOW	0	0	2.50	1.21	$10^5 M^3$/SEC
ARGUMENTS					
x_1 =	2.1779	2.7750	1.1285	1.4395	
x_2 =	.5481	1.2435	.2839	.6440	
x_3 =	2.8735	2.5331	1.4900	1.3131	
x_4 =	1.4704	.9884	.7622	.5123	
$K_5 l_5$ =	69.2°		35.9°		
x_6 =	.3582	.6741	.1857	.3495	

APPENDIX 2

Part 2

CURRENTS FOR THE 6 CHANNEL MODEL

Junction	WITHOUT TRANSPORT		WITH TRANSPORT		UNITS
	M 2	K 1	M 2	K 1	
0	-3.3	.6	-9.7	.3	
1	-1.8	1.7	-3.8	1.1	
2	1.6	17.0	10.0	12.0	
3	21.0	18.0	16.0	13.0	
4	6.8	4.9	7.0	3.6	
5	1.8	.6	5.2	.8	
6	0	0	24.0	2.3	CM/SEC
TRANSPORT AT THE HEAD	0	0	17.5	1.7	$10^6 M^3$/SEC

These values are obtained assuming the boundary value Z_o at the mouth to be 70 cm and 14 cm for M_2 and K_1 respectively.

0 denotes the mouth of the sea; 1, the junction of channels 1 and 2, etc.

APPENDIX 2

Part 3

COMPARISON BETWEEN THE TRANSPORT bhv FOR THE 6 CHANNEL MODEL AND THAT FOR THE THREE BASIN MODEL

y	6 Channel	3 Basin
0	-24.4	- 9.25 ⟶ - 11.9
6	-4.86	- 2.79 ⟶ - 5.12
10	1.30	2.95 ⟶ ∞
11.4	1.92	
12		3.93 ⟶ ∞
14	2.97	4.21 ⟶ 4.39
22	2.21	2.93 ⟶ 3.35
23.5 x 10^2 km	1.74	2.03 ⟶ 2.50 x 10^6 m^3 /sec

APPENDIX 3

CONSTANTS

The upper number refers to M_2 ; the lower one to K_1.

$$\sigma = \begin{pmatrix} 1.4059 \\ .7292 \end{pmatrix} \text{x}10^{-4} \text{ sec}^{-1} \qquad g = 981 \text{ cm/sec}^2$$

$$2a_1 = 7.62\text{x}10^7 \text{ cm} \qquad 2a_2 = \begin{pmatrix} 3.54 \\ 1.77 \end{pmatrix} \text{x}10^7 \text{ cm} \qquad 2a_3 = 4.39\text{x}10^7 \text{ cm}$$

$$h_1 = 1.57\text{x}10^5 \text{ cm} \qquad h_2 = 3.7\text{x}10^5 \text{ cm} \qquad h_3 = 8.0\text{x}10^5 \text{ cm}$$

$$2L_1 = 10\text{x}10^7 \text{ cm} \qquad 2L_2 = 2\text{x}10^7 \text{ cm} \qquad 2L_3 = \begin{pmatrix} 14.5 \\ 11.5 \end{pmatrix} \text{x}10^7 \text{ cm}$$

$$2\omega_1 = 1.2913\text{x}10^{-4} \text{ sec}^{-1} \qquad 2\omega_2 = 1.3486\text{x}10^{-4} \text{ sec}^{-1} \qquad 2\omega_3 = 1.3955\text{x}10^{-4} \text{ sec}^{-1}$$

$$\lambda_1 = .2525 \qquad \lambda_2 = \begin{pmatrix} .2527 \\ .1264 \end{pmatrix} \qquad \lambda_3 = .2200$$

$$\lambda_1' = 1.0411\text{x}10^{-8}\text{cm}^{-1} \qquad \lambda_2' = 2.2424\text{x}10^{-8}\text{cm}^{-1} \qquad \lambda_3' = 1.5743\text{x}10^{-8}\text{cm}^{-1}$$

$$\lambda_1' a_1 = .3967 \qquad \lambda_2' a_2 = \begin{pmatrix} .3969 \\ .1985 \end{pmatrix} \qquad \lambda_3' a_3 = .3456$$

$$K_o^{\textcircled{1}\prime} = \begin{pmatrix} 1.1335 \\ .5875 \end{pmatrix} \text{x}10^{-8}\text{cm}^{-1} \qquad K_o^{\textcircled{2}\prime} = \begin{pmatrix} 2.3377 \\ 1.2118 \end{pmatrix} \text{x}10^{-8}\text{cm}^{-1} \qquad K_o^{\textcircled{3}\prime} = \begin{pmatrix} 1.5861 \\ .8222 \end{pmatrix} \text{x}10^{-8}\text{cm}$$

$$2K_o^{\textcircled{1}} l_1 = \begin{pmatrix} 1.1335 \\ .5876 \end{pmatrix} \text{ rad} \qquad 2K_o^{\textcircled{2}} l_2 = \begin{pmatrix} .4675 \\ .2424 \end{pmatrix} \text{ rad} \qquad 2K_o^{\textcircled{3}} l_3 = \begin{pmatrix} 2.2998 \\ .9455 \end{pmatrix} \text{ rad}$$

$$\alpha_1 = 2.4725 \text{x} 10^4 \text{ sec}^2 \qquad \alpha_2 = \begin{pmatrix} 1.1486 \\ .5743 \end{pmatrix} \text{x} 10^4 \text{ sec}^2 \qquad \alpha_3 = 1.4244 \text{x} 10^4 \text{sec}^2$$

$$2\omega_1 \alpha_1 = 3.1927 \text{ sec} \qquad 2\omega_2 \alpha_2 = \begin{pmatrix} 1.5490 \\ .7745 \end{pmatrix} \text{sec} \qquad 2\omega_3 \alpha_3 = 1.9878 \text{ sec}$$

$$\sigma \alpha_1 = \begin{pmatrix} 3.4761 \\ 1.8029 \end{pmatrix} \text{sec} \qquad \sigma \alpha_2 = \begin{pmatrix} 1.6148 \\ .4188 \end{pmatrix} \text{ sec} \qquad \sigma \alpha_3 = \begin{pmatrix} 2.0026 \\ 1.0386 \end{pmatrix} \text{ sec}$$

$$(h_1/g)^{\frac{1}{2}} = 12.6433 \text{ sec} \qquad (h_2/g)^{\frac{1}{2}} = 6.1301 \text{ sec} \qquad (h_3/g)^{\frac{1}{2}} = 9.0355 \text{ sec}$$

$$F^{\textcircled{1}} = .9746$$

$$G^{\textcircled{1}} = \begin{pmatrix} .1095 \\ .0537 \end{pmatrix} \qquad G^{\textcircled{3}} = \begin{pmatrix} .3753 \\ .1841 \end{pmatrix}$$

Note: All the calculations have been carried through in cgs although many values of length or volume are quoted in other units in the body of the thesis for easier apprehension.